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On the Noise-Induced Passage through an Unstable Periodic Orbit I: Two-Level Model

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We consider the problem of stochastic exit from a planar domain, whose boundary is an unstable periodic orbit, and which contains a stable periodic orbit. This problem arises when investigating the distribution of noise-induced phase slips between synchronized oscillators, or when studying stochastic resonance far from the adiabatic limit. We introduce a simple, piecewise linear model equation, for which the distribution of first-passage times can be precisely computed. In particular, we obtain a quantitative description of the phenomenon of cycling: The distribution of first-passage times rotates around the unstable orbit, periodically in the logarithm of the noise intensity, and thus does not converge in the zero-noise limit. We compute explicitly the cycling profile, which is universal in the sense that it depends only on the product of the period of the unstable orbit with its Lyapunov exponent.

KEY WORDS: Stochastic exit problem; diffusion exit; first-exit time; large deviations; metastability; level-crossing problem; limit cycle; synchronization; phase slip; cycling; stochastic resonance.

1. INTRODUCTION

One of the remarkable effects of additive noise on a deterministic dynamical system is to enable transitions between otherwise isolated attractors. This phenomenon, known as *activation* in physics and chemistry, is at the origin of the so-called *stochastic exit problem*. Consider a stochastic differential equation (SDE) of the form

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{1.1}$$

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Given an attractor $\mathscr{A}$ of the deterministic system $\dot{x} = f(x)$ and a domain $\mathscr{D}$ containing $\mathscr{A}$ (usually $\mathscr{D}$ is taken to be positively invariant under the deterministic flow), the exit problem consists in characterizing

- the distribution of the random time $\tau = \inf\{t > 0 : x_t \notin \mathscr{D}\}$ at which paths of the SDE, starting in some initial point in $\mathscr{D}$, leave the domain;
- the distribution of the exit location x_τ on the boundary $\partial\mathscr{D}$ of the domain.

The best understood situation is the one where $f = -\nabla U$ derives from a potential U, and the attractor $\mathscr{A}$ is simply the bottom $x^\star$ of a potential well. Assume that $\mathscr{D}$ contains no other equilibrium point in its interior, and a (non-degenerate) saddle point x_1 of U on its boundary. The expectation of the first-exit time τ is then given by

$$\mathbb{E}\{\tau\} = c(\sigma)\, \mathrm{e}^{[U(x_1) - U(x^\star)]/2\sigma^2}. \tag{1.2}$$

The form of the exponent was already known to Arrhenius,[1] while the value of the prefactor $c(\sigma)$, which depends on the curvature of U at $x^\star$ and x_1, was determined by Eyring and Kramers in the one-dimensional case.[14, 20]

Many refinements of this result exist. On the one hand, the theory of large deviations allows to compute the exponent in a much more general setting, with a drift term not necessarily deriving from a potential.[18] In this case, U is replaced by the notion of a *quasipotential*. On the other hand, the computation of the prefactor $c(\sigma)$ in n dimensions has been addressed using methods of perturbation theory (see, for instance, refs. 2 and 15). Precise results on the relation between the expected first-exit time, capacities and the small eigenvalues of the generator of the diffusion have been obtained recently for drift coefficients deriving from a potential.[3, 6]

The distribution of τ approaches an exponential one in the small-noise limit,[6, 8] and the exit location x_τ is strongly concentrated near the saddle x_1.[18] Another related property is that for a fixed time $t \ll \mathbb{E}\{\tau\}$, the probability to leave $\mathscr{D}$ before time t is exponentially small, of the order $t/\mathbb{E}\{\tau\}$. This phenomenon is known as *metastability*, because the state x_t may spend extremely long time spans near local minima of the potential, while the stationary density has most of its mass concentrated near the global potential minima.

A more difficult situation arises, in two-dimensional systems, when the boundary of $\mathscr{D}$ is an unstable periodic orbit of the deterministic flow. In that case, large deviations theory cannot be applied directly, because $\partial\mathscr{D}$ is a so-called characteristic boundary. The situation where $\partial\mathscr{D}$ is an unstable periodic orbit and $\mathscr{A}$ an equilibrium point has been considered by Day[9, 10] and by Maier and Stein.[21] Day[10] proved that the distribution of the first-exit location x_τ displays a striking behaviour, called *cycling*: As the noise intensity σ decreases, the density of x_τ rotates around the boundary, as a function of $|\log \sigma|$. Using the concept of the most probable exit path (MPEP) and WKB approximations, Maier and Stein also found a periodic dependence on $|\log \sigma|$ for the rate of escape per unit time through $\partial\mathcal{D}$. The intuition behind their estimates is that the MPEP spirals geometrically towards $\partial\mathcal{D}$, until it reaches a neighbourhood of order σ of $\partial\mathcal{D}$ and escape becomes likely. Where and when this happens depends periodically on $|\log \sigma|$.

A related situation, which has not yet been analysed in detail, arises when the boundary of $\mathcal{D}$ is an unstable periodic orbit but $\mathcal{A}$ is a stable periodic orbit. This situation is important in applications:

1. When studying the dynamics of two coupled, slightly different phase oscillators, the onset of *synchronization* is known to correspond to a saddle-node bifurcation of periodic orbits. Below a threshold coupling strength, the motion is typically quasiperiodic, meaning that the oscillators are not synchronized. Above this threshold, a pair of periodic orbits of opposite stability appears, the stable one corresponding to a synchronized state in which the phase difference between the oscillators may vary periodically but is bounded (see, for instance, ref. 22).

   Adding noise to the system will cause *phase slips* to occur, in which the phase difference changes by 2π. Such a phase slip necessarily involves crossing the unstable periodic orbit, so that the determination of the distribution of phase slips requires the determination of the distribution of first-passage times at an unstable orbit.

2. The phenomenon of *stochastic resonance* occurs for instance when a double-well potential is forced periodically in time.[7] Noise-induced transitions between potential wells are more favourable from the shallower to the deeper well, when the barrier between them is lowest. As a result, typical paths of the system will contain a periodic component, see, for instance, refs. 16, 19, and 25.

   On the mathematically rigorous level, the situation is relatively well understood in the adiabatic case, that is for slow forcing, when the paths spend most of the time near the bottom of a potential well.[4, 17] When the forcing is not slow, however, it is easy to see that the deterministic system still admits three periodic solutions, two stable ones oscillating around the potential wells, and an unstable one oscillating around the saddle. Hence the investigation of transition between wells again involves the understanding of first-passage times at the unstable orbit.

One may point out that in the case of synchronization, the phase space has the topology of a torus, while for stochastic resonance, it has the topology of a cylinder. However, both situations have in common the fact that a path, starting near a stable periodic orbit, has to cross an unstable periodic orbit for a transition to become possible.

In the present work, we focus on the dynamics of the paths up to their first crossing of $\partial\mathcal{D}$. Our aim is to give a precise characterization of the distribution of exit times and locations, in particular in the metastable regime. In order to highlight the mechanism responsible for cycling, we concentrate here on a simplified, piecewise linear model equation, which can be solved exactly to leading order. The general case will be discussed in a forthcoming publication.[5]

In our model, we neglect the diffusion in the longitudinal direction, so that the exit location is determined by the exit time, modulo the period of the orbit. The main result, Theorem 2.3, gives an explicit expression for the density of the first-exit time. If T is the period of the unstable orbit, and λ its Lyapunov exponent, for a large range of metastable times t such that $2\,|\log\sigma| \ll \lambda t \ll \mathrm{e}^{\text{const}/\sigma^2}$, this density is given by

$$p_+(t) \simeq \text{const}\,\sigma\theta'(t)\,P\left(\frac{|\log\sigma| - \theta(t)}{\lambda T}\right)\mathrm{e}^{-R^2/2\sigma^2}. \tag{1.3}$$

Here R^2 describes the exponential rate of escape provided by large deviations theory. The *cycling profile* $P(x)$ is an explicitly known, universal function, depending only on λT. The model-dependent intrinsic time $\theta(t)$, which satisfies $\theta(t+T) = \theta(t) + \lambda T$, describes the "velocity" $1/\theta'(t)$ with which the cycling profile rotates around the orbit.

The regime $\lambda t \leqslant 2\,|\log\sigma|$ is transient, in the sense that paths have not yet reached their typical spreading, and thus $p_+(t)$ is smaller than (1.3) by a factor $\mathrm{e}^{-\text{const}\,\mathrm{e}^{-\lambda t}/\sigma^2}$. This initial phase is not observed in ref. 21, where the authors artificially create a stationary regime by reinjecting escaped paths into the attractor.

Though we do not treat in detail the asymptotic regime $t \gg \mathrm{e}^{\text{const}/\sigma^2}$, results by Day[10] imply that the superposition $\sum_{k\geqslant 0} p_+(t+kT)$, which does not take into account the winding number k of paths around the unstable orbit, has a similar behaviour as (1.3). In fact, we expect (1.3) to hold for times larger than $\mathrm{e}^{\text{const}/\sigma^2}$, with an additional factor, slowly decaying like $\exp\{-t\mathrm{e}^{-R^2/2\sigma^2}\}$.

2. MODEL AND RESULTS

2.1. Periodic Orbits and Coordinate Systems

Consider a two-dimensional ordinary differential equation $\dot{x} = f(x)$ admitting a stable periodic orbit enclosed by an unstable one. We assume that the domain $\mathcal{S}$ lying between the orbits has the topology of an annulus and contains no invariant sets. In that case, one can choose polar-like coordinates (r, φ), such that φ is 2π-periodic and $\dot{\varphi} > 0$ in a neighbourhood of $\mathcal{S}$.

In the deterministic case, it is customary to use φ as new independent variable, to obtain a one-dimensional non-autonomous system

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = f_r(r, \varphi). \tag{2.1}$$

Furthermore, it is possible to choose r in such a way that the stable periodic orbit corresponds to $r = -1$, and the unstable one to $r = +1$.

Example 2.1. Synchronization. The onset of synchronization between two weakly coupled phase oscillators can be viewed as a saddle-node bifurcation of periodic orbits. A normal-form analysis shows that close to the bifurcation point, the dynamics is governed to leading order by an equation of the form

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = r^2 - \varepsilon c(\varphi), \tag{2.2}$$

where ε is the bifurcation parameter. If $c(\varphi) > 0$ for all φ, a straightforward analysis of the Poincaré map shows the existence, for $\varepsilon > 0$, of two periodic orbits $r = r_+(\varphi)$ and $r = r_-(\varphi)$ of opposite stability, separated by a distance of order $\sqrt{\varepsilon}$. The linear transformation

$$r = \frac{r_+(\varphi) + r_-(\varphi)}{2} + \frac{r_+(\varphi) - r_-(\varphi)}{2}\, y \tag{2.3}$$

yields the equation

$$\frac{\mathrm{d}y}{\mathrm{d}\varphi} = \frac{1}{2}\left[r_+(\varphi) - r_-(\varphi)\right](y^2 - 1). \tag{2.4}$$

The periodic orbits are now located in $y = \pm 1$. Note that the stable (unstable) orbit is attracting (repelling) more strongly for those values of φ for which the orbits in the original system are further apart.

When noise is added to the system, it will in general affect both the transversal r- and the longitudinal φ-direction, so that one is led to analyse the system

$$\begin{aligned} \mathrm{d}r_t &= f_r(r_t, \varphi_t)\, \mathrm{d}t + \sigma g_r(r_t, \varphi_t)\, \mathrm{d}W_t, \\ \mathrm{d}\varphi_t &= f_\varphi(r_t, \varphi_t)\, \mathrm{d}t + \sigma g_\varphi(r_t, \varphi_t)\, \mathrm{d}W_t, \end{aligned} \tag{2.5}$$

possibly with $f_\varphi = 1$. Here we consider σ as a small parameter controlling the noise intensity, while g_r and g_φ are fixed functions (of order one).

Example 2.2. Stochastic Resonance. A classical example of a system showing stochastic resonance consists of an overdamped particle in a periodically forced double-well potential, perturbed by additive noise. For a Ginzburg–Landau potential, the equation reads

$$\mathrm{d}r_t = [r_t - r_t^3 + A\cos(2\pi t/T)]\,\mathrm{d}t + \sigma \mathrm{d}W_t. \tag{2.6}$$

Again, using a Poincaré section, one easily proves the existence of one unstable and two stable periodic orbits (for any value of the period T). If $r_-(t)$ denotes, say, the lower stable orbit, and $r_+(t)$ the unstable one, a transformation of the form (2.3) yields Eq. (2.5) with $f_\varphi = 1$, $g_\varphi = 0$ (that is, $\varphi = t$), and g_r depending only on φ.

The general case of Eq. (2.5) will be discussed in a forthcoming work.[5] Here we shall concentrate on a simplified model which focuses on the main mechanism responsible for the oscillatory behaviour of the first-exit time.

2.2. Simplified Two-Level Model

The main motivation to introduce the two-level model is the fact that the distribution of first-exit times will be mainly determined by the dynamics near the unstable periodic orbit. The dynamics in the remaining phase space can thus be modelled by the simplest possible equation, that is, a linear one. More precisely, we will simplify (2.5) by

- neglecting the term g_φ, whose effect is a slow diffusion on a time scale $1/\sigma^2$;
- neglecting the r-dependence of g_r, which is not important near the unstable orbit;
- replacing f_r by a piecewise linear function of r.

With these approximations, we arrive at the system

$$\mathrm{d}y_t = f(y_t, t)\,\mathrm{d}t + \sigma g(t)\,\mathrm{d}W_t, \tag{2.7}$$

where f and g are periodic in t, and we may assume that $f(\pm 1, t) = 0$ for all t. Here time t is identified with the angle φ lifted to the real axis.

In order to avoid certain technical difficulties when dealing with the stochastic process, we will actually switch between two linear equations defined in slightly overlapping regions. These equations are

$$\begin{aligned} \mathrm{d}y_t^- &= -a(t)(y_t^- + 1)\,\mathrm{d}t + \sigma g(t)\,\mathrm{d}W_t, \\ \mathrm{d}y_t^+ &= a(t)(y_t^+ - 1)\,\mathrm{d}t + \sigma g(t)\,\mathrm{d}W_t, \end{aligned} \tag{2.8}$$

where $a(t)$ and $g(t)$ are T-periodic, positive functions, which are bounded away from zero (detailed assumptions will be given in Section 2.3 below). We denote by $y_t^-(t_0, y_0)$ and $y_t^+(t_0, y_0)$ the solutions of these equations with initial conditions $y_{t_0}^- = y_0$ or $y_{t_0}^+ = y_0$, respectively. The stable orbit located at $y = -1$ and the unstable orbit at $y = +1$ have Lyapunov exponents $\mp\lambda$, where

$$\lambda = \frac{\alpha(T)}{T}, \qquad \text{with} \quad \alpha(t) = \int_0^t a(s)\,\mathrm{d}s. \tag{2.9}$$

The switching between the processes occurs upon reaching levels $1-\delta_1 \in (0,1)$ from below and $1-\delta_2 \in (0, 1-\delta_1)$ from above (see Fig. 1). More precisely, consider the stopping times

$$\begin{aligned} \tau_\nearrow &= \tau_\nearrow(t_0, y_0) = \inf\{t > t_0 : y_t^-(t_0, y_0) > 1-\delta_1\} \in [t_0, \infty], \\ \tau_\searrow &= \tau_\searrow(t_0, y_0) = \inf\{t > t_0 : y_t^+(t_0, y_0) < 1-\delta_2\} \in [t_0, \infty]. \end{aligned} \tag{2.10}$$

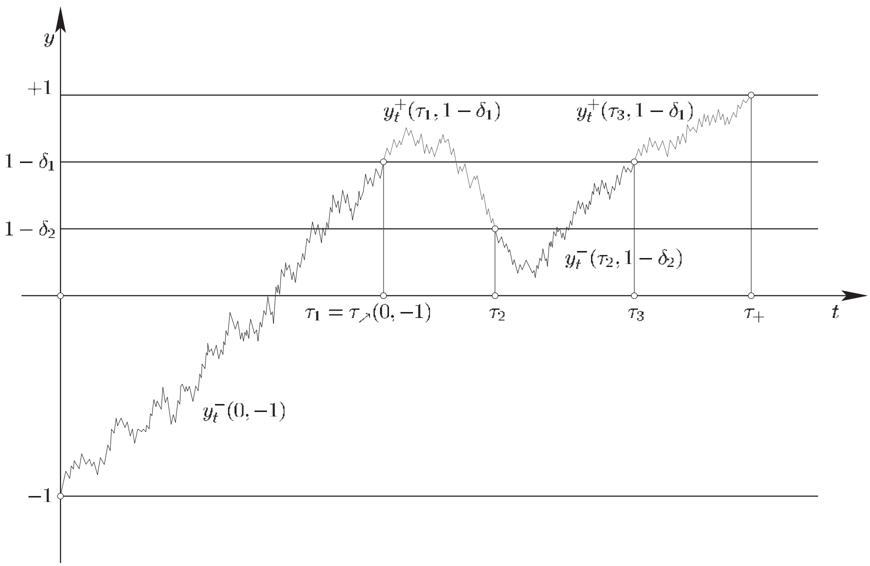

Fig. 1. The process y_t is defined by switching between two linear processes y_t^- and y_t^+ each time either the level $1-\delta_1$ is crossed from below or the level $1-\delta_2$ is crossed from above.

Then the process $\{y_t\}_{t\geqslant 0}$ is defined in the following way:

$$y_t = \begin{cases} y_t^-(0,-1) & \text{for} \quad 0 \leqslant t \leqslant \tau_1 = \tau_{\nearrow}(0,-1), \\ y_t^+(\tau_1, 1-\delta_1) & \text{for} \quad \tau_1 \leqslant t \leqslant \tau_2 = \tau_{\searrow}(\tau_1, 1-\delta_1), \\ y_t^-(\tau_2, 1-\delta_2) & \text{for} \quad \tau_2 \leqslant t \leqslant \tau_3 = \tau_{\nearrow}(\tau_2, 1-\delta_2), \end{cases} \tag{2.11}$$

and so on. We are interested in determining the distribution of the first-passage time at the unstable orbit, namely in the distribution of

$$\tau_+ = \inf\{t > 0 : y_t > +1\}. \tag{2.12}$$

We use the notation

$$p_+(t) = \frac{\partial}{\partial t} \mathbb{P}^{0,-1}\{\tau_+ \leqslant t\} \tag{2.13}$$

for the density of τ_+, where the superscript in $\mathbb{P}^{0,-1}\{\cdot\}$ indicates that we start at time $t=0$ on the stable orbit in $y=-1$. For brevity, we shall call $p_+(t)$ the first-passage density of y_t to $+1$.

Note that since φ is proportional to $t \pmod T$, the first-exit time τ_+ directly allows to determine the first-exit location φ_{τ_+}. However, τ_+ actually contains more information since it also keeps track of the number of revolutions (or winding number) of the path around the stable orbit.

In the physics literature, one often considers the rate of escape per unit time, defined as minus the time-derivative of the probability to be inside the domain $\mathscr{D}$. The difference between rate of escape and first-passage density is that the former counts negatively the paths which have left $\mathscr{D}$ before time t, but returned into $\mathscr{D}$ by time t, while these paths are not counted by the first-passage density. If the system is symmetric with respect to the boundary $\mathscr{D}$, the reflection principle implies that the rate of escape is equal to half the first-passage density.

2.3. Main Results

Four periodic functions will play an important role in the statement of the results. They are given by

$$v_-^{\text{per}}(t) = \frac{1}{1-\mathrm{e}^{-2\lambda T}} \int_t^{t+T} \mathrm{e}^{-2\alpha(t+T,s)} g(s)^2 \, \mathrm{d}s, \tag{2.14}$$

$$\hat{v}_+^{\text{per}}(t) = \frac{1}{\mathrm{e}^{2\lambda T}-1} \int_t^{t+T} \mathrm{e}^{2\alpha(t+T,s)} g(s)^2 \, \mathrm{d}s, \tag{2.15}$$

$$\rho^{\text{per}}(t)^2 = \frac{\delta_1^2}{\hat{v}_+^{\text{per}}(t)} + \frac{(2-\delta_1)^2}{v_-^{\text{per}}(t)}, \tag{2.16}$$

$$v^\star(t) = \frac{g(t)^2}{2a(t)}, \tag{2.17}$$

where $\alpha(t,s) = \alpha(t) - \alpha(s)$. The first two functions are directly related to the variances of $y_t^-(t_0, y_0)$ and $y_t^+(t_0, y_0)$. These are independent of the initial condition y_0, and have respective values

$$\sigma^2 v_-(t,t_0) = \sigma^2[v_-^{\text{per}}(t) - \mathrm{e}^{-2\alpha(t,t_0)} v_-^{\text{per}}(t_0)], \tag{2.18}$$

$$\sigma^2 v_+(t,t_0) = \sigma^2[\mathrm{e}^{2\alpha(t,t_0)} \hat{v}_+^{\text{per}}(t_0) - \hat{v}_+^{\text{per}}(t)]. \tag{2.19}$$

The function $v^\star(t)$ allows to determine the qualitative behaviour of $v_-^{\text{per}}(t)$ and $\hat{v}_+^{\text{per}}(t)$. Indeed, since $v_-^{\text{per}}(t)$ satisfies the differential equation $\dot{v} = -2a(t)\,v + g(t)^2$, it is increasing whenever it lies below $v^\star(t)$ and decreasing whenever it lies above. Similarly, $\hat{v}_+^{\text{per}}(t)$ is decreasing when it lies below $v^\star(t)$ and increasing when it lies above (Fig. 2). This shows in particular that $v_-^{\text{per}}(t)$ and $\hat{v}_+^{\text{per}}(t)$ always lie between the maximum $\bar{v}$ and the minimum $\underline{v}$ of $v^\star(t)$, and that they reach their extremal values when crossing $v^\star(t)$. One can also see from the graphical representation that smaller values of the period T lead to smaller amplitudes of $v_-^{\text{per}}(t)$ and $\hat{v}_+^{\text{per}}(t)$.

Our main assumptions are the following:

Hypotheses

H1. The functions $a(t)$ and $g(t)$ are twice continuously differentiable, positive, T-periodic and bounded away from zero.

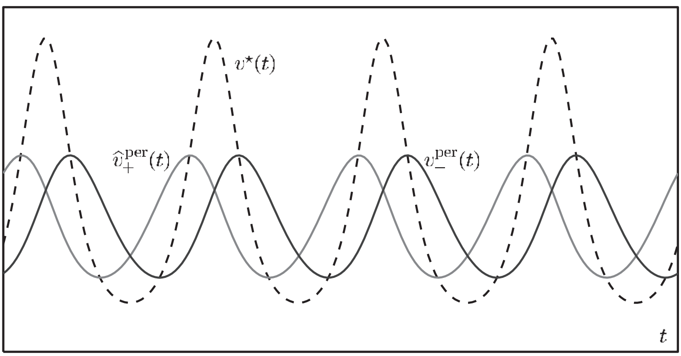

Fig. 2. The periodic functions $v^\star(t)$, $v_-^{\text{per}}(t)$ and $\hat{v}_+^{\text{per}}(t)$.

H2. $v^\star(t)$ has exactly one maximum and one minimum in $[0, T)$, with values $\bar{v} > \underline{v} > 0$.

H3. There is a constant $\Delta > 0$ such that $v_-^{\text{per}}(t) \leqslant 2v^\star(t)(1-\Delta)$ and $\hat{v}_+^{\text{per}}(t) \leqslant 2v^\star(t)(1-\Delta)$ for all t.

H4. $\delta_2/(2-\delta_2) \leqslant \sqrt{\underline{v}/\bar{v}}$.

H5. $\rho^{\text{per}}(t)$ has exactly one minimum in $[0, T)$ at a time $t = s^\star$, which is quadratic. We set

$$R = \rho^{\text{per}}(s^\star) = \inf_{t \in [0,T)} \rho^{\text{per}}(t). \tag{2.20}$$

Note that Hypothesis H3 amounts to requiring that the most probable exit paths of the processes y_t^- and y_t^+ can cross at most once the levels $1-\delta_1$ and $1-\delta_2$ (see the Appendix). It is always satisfied when T is large enough, because then the variances v_-^{per} and $\hat{v}_+^{\text{per}}$ track $v^\star$ adiabatically. Is is also satisfied if $\bar{v} < 2\underline{v}(1-\Delta)$. Hypothesis H4 ensures that y_t^- is dominated by y_t^+ near the unstable orbit. We make Hypothesis H5 mainly in order to simplify the presentation, it is in fact sufficient to require that the deepest minimum of $\rho^{\text{per}}(t)$ be quadratic, which is generically true if $v^\star(t)$ is non-trivial.

Under Assumptions H1–H5, the following result on the first-passage density $p_+(t)$ holds whenever σ is small enough.

Theorem 2.3. There exists a $\sigma_0 > 0$ such that for all $\sigma \leqslant \sigma_0$, the density $p_+(t)$ of τ_+ is given by

$$p_+(t) = c(t, \sigma)\, \mathrm{e}^{-R^2/2\sigma^2}, \tag{2.21}$$

where R is defined in (2.20) and the prefactor $c(t, \sigma)$ depends on t and σ in the following way:

1. **Transient Regime:** If $0 \leqslant \alpha(t) < 2\,|\log \sigma|$, then

$$c(t, \sigma) \leqslant \text{const}\, \frac{1}{\sigma^2} \exp\left\{ -\frac{L\mathrm{e}^{-\alpha(t)}}{\sigma^2} \right\}, \tag{2.22}$$

where L is a positive constant.

2. **Metastable Regime:** Let $\Delta_0 = \Delta/(1-\Delta) \wedge (\delta_2 - \delta_1)/\delta_1 \wedge 1$.[3] There exists a constant $\beta > 0$ such that for t satisfying $2\,|\log \sigma| \leqslant \alpha(t) \ll \sigma^3 \mathrm{e}^{\beta\Delta_0^2/2\sigma^2}$,

$$c(t, \sigma) = \sigma C_0(s^\star)\, \theta'(t)\, P\left(\frac{|\log \sigma| - \theta(t)}{\lambda T} \right) [1 + r(\sigma)], \tag{2.23}$$

[3] We write $a \wedge b$ to denote the minimum of two real numbers a and b.

where $P(x) > 0$ is an explicitly known periodic function with period 1, see Eqs. (2.28) and (2.29) below, and

$$\theta(t) = \alpha(t, s^\star) - \frac{1}{2}\log \hat{v}_+^{\mathrm{per}}(t) - \log \frac{\delta_1}{\hat{v}_+^{\mathrm{per}}(s^\star)}, \tag{2.24}$$

$$\theta'(t) = \frac{1}{2}\frac{g(t)^2}{\hat{v}_+^{\mathrm{per}}(t)}. \tag{2.25}$$

$C_0(s^\star)$ is a constant given by

$$C_0(s^\star) = 4\,\frac{2-\delta_1}{\delta_1}\,\frac{g(s^\star)^2}{\sqrt{\pi\partial_{ss}(\rho^{\mathrm{per}}(s^\star)^2)}}\,\frac{\hat{v}_+^{\mathrm{per}}(s^\star)^{1/2}}{v_-^{\mathrm{per}}(s^\star)^{3/2}}\left[1-\frac{v_-^{\mathrm{per}}(s^\star)}{2v^\star(s^\star)}\right], \tag{2.26}$$

and the error term satisfies

$$r(\sigma) = \mathcal{O}\left(\sigma + \frac{1}{\sigma^2}\,\mathrm{e}^{-\alpha(t)}\right). \tag{2.27}$$

3. **Asymptotic Regime:** If $\alpha(t) \geqslant \mathrm{const}\,\mathrm{e}^{R/2\sigma^2}$, most paths will have crossed the unstable orbit as least once, and thus the density decays.

The periodic function $P(x)$ appearing in (2.23) has the following expressions:

$$P(x) = \sum_{\ell=-\infty}^{\infty} A(\lambda T(\ell - x)), \qquad \text{with} \quad A(x) = \frac{1}{2}\,\mathrm{e}^{-2x}\exp\left\{-\frac{1}{2}\,\mathrm{e}^{-2x}\right\}, \tag{2.28}$$

which is particularly useful for large T, and the Fourier series

$$P(x) = \sum_{q=-\infty}^{\infty} \hat{P}(q)\,\mathrm{e}^{2\pi i q x}, \qquad \text{with} \quad \hat{P}(q) = \frac{1}{2\lambda T}\,\frac{1}{2^{\pi i q/\lambda T}}\,\Gamma\left(1 - \frac{\pi i q}{\lambda T}\right), \tag{2.29}$$

where Γ is the Euler Gamma function. This series converges quickly when T is small.

Before discussing the implications of this result, let us briefly sketch the proof, the details of which are given in Sections 3 to 5.

A first step is to determine the density $\psi_-(s, 0)$ of the first-passage time at $1-\delta_1$, when the first switching occurs. We will show in Proposition 4.1 that it is given by

$$\psi_-(s, 0) = \frac{1}{\sigma}\,c_-(s, 0)\,\mathrm{e}^{-(2-\delta_1)^2/2\sigma^2 v_-(s,0)}, \tag{2.30}$$

where $\sigma^2 v_-(s,0)$ is the variance of y_s^-, see (2.18), and the prefactor $c_-(s,0)$ does not play an important role. Since $v_-(s,0)$ behaves asymptotically like $v_-^{\mathrm{per}}(s)$, the first-passage times at $1-\delta_1$ are sharply concentrated in small neighbourhoods of the local maxima of $v_-^{\mathrm{per}}(s)$. All these maxima correspond to a single point in space, but with a different number of revolutions around the stable orbit. In fact, the theory of large deviations allows to establish a qualitatively similar behaviour in the general, nonlinear case.

Next we consider the density $t \mapsto q(t,s)$ of paths starting at time s in $1-\delta_1$ and reaching the unstable orbit in $+1$ at time t. In Sections 4.2 and 4.3, we establish the expression

$$q(t,s) = \frac{1}{\sigma}\,\bar{c}_+(t,s)\,\mathrm{e}^{-\delta_1^2/2\sigma^2\hat{v}_+(t,s)}, \tag{2.31}$$

where $\bar{c}_+(t,s)$ decays like $\mathrm{e}^{-2\alpha(t,s)}$ and $\hat{v}_+(t,s) = \mathrm{e}^{-2\alpha(t,s)} v_+(t,s)$. The exponential decay of the prefactor is due to the fact that a large, asymptotically constant fraction of paths leave the neighbourhood of the unstable orbit during each period. When computing $q(t,s)$, we have to take into account all paths crossing the levels $1-\delta_1$ and $1-\delta_2$ arbitrarily often, before reaching $+1$. This is done with the help of a renewal equation, discussed in Section 3. The main contribution, however, comes from paths going directly from $1-\delta_1$ to $+1$, without returning to $1-\delta_2$.

For $t \gg s$, $\hat{v}_+(t,s)$ approaches the function $\hat{v}_+^{\mathrm{per}}(s)$, which is periodic in the *initial* time s. This means that paths reaching $+1$ at some fixed time t have left $1-\delta_1$ with approximately equal probability near any local maximum of $\hat{v}_+^{\mathrm{per}}(s)$. This *bottleneck effect* is due to the fact that the level $1-\delta_1$ corresponds to a noncharacteristic boundary where large-deviation results guarantee concentration of paths, while the unstable orbit causes a strong dispersion of paths.

The first-passage density $p_+(t)$ is given by the integral of $q(t,s)\,\psi_-(s,0)$, which can be evaluated by the Laplace method, yielding a sum over all minima of the function

$$s \mapsto \frac{\delta_1^2}{\hat{v}_+(t,s)} + \frac{(2-\delta_1)^2}{v_-(s,0)}. \tag{2.32}$$

For $0 \ll s \ll t$, this function is close to $\rho^{\mathrm{per}}(s)^2$, and thus has one minimum per period. A careful analysis, given in Section 5.1, shows that for t near nT, the kth term of the sum is proportional to

$$\sigma\left[\frac{\gamma(t)}{2\sigma^2}\exp\left\{-2(n-k)\,\lambda T - \frac{\gamma(t)}{2\sigma^2}\,\mathrm{e}^{-2(n-k)\,\lambda T}\right\}\right], \tag{2.33}$$

for some $\gamma(t)$ (the same expression is obtained in ref. 21, using WKB approximations). This term is the contribution of the paths making k revolutions around the stable orbit before reaching $1-\delta_1$, and $n-k$ revolutions afterwards. Adding λT to $|\log \sigma|$ will multiply σ^2 by $e^{-2\lambda T}$, but this only results in a rearrangement of the terms in the sum, without changing its value up to small boundary terms. This roughly explains why the prefactor is periodic in $|\log \sigma|$. A more detailed analysis of the sum, given in Section 5.2, is necessary to obtain the precise form (2.23).

2.4. Discussion

Exponential Asymptotics

The exponential rate R^2 occurring in (2.21) is independent of t, meaning that on the level of exponential asymptotics, all points on the unstable orbit are reached with the same probability. This is in sharp contrast with, say, points on the intermediate level $1-\delta_1$, which are reached with nonconstant exponential rate $(2-\delta_1)^2/2v_-(t,0)$. This difference is a natural consequence of the fact that the unstable orbit is a characteristic boundary, unlike the level $1-\delta_1$. The rate $R^2/2$ is the value of the so-called *boundary quasipotential* of the Wentzell–Freidlin theory.

Note that the expressions (2.21) and (2.23) for the first-passage density are not in contradiction with the property

$$\lim_{\sigma \to 0} \mathbb{P}\{e^{(R^2-\delta)/2\sigma^2} \leqslant \tau_+ \leqslant e^{(R^2+\delta)/2\sigma^2}\} = 1, \tag{2.34}$$

holding according to the Wentzell–Freidlin theory for any fixed $\delta > 0$. This expression suggests that first-passage times are concentrated near Kramers' time $e^{R^2/2\sigma^2}$, but the fact that δ is independent of σ actually allows for a large spreading of the distribution of τ_+ which manifests itself in the behaviour of the prefactor $c(t,s)$.

Time Scales

Two time scales play a role for the behaviour of the first-passage density $p_+(t)$. They delimit three time domains, whose boundaries, however, are not particularly sharp. The time scale $t_{\rm r} = 2\,|\log \sigma|/\lambda$ is the time needed for the variances v_- and $\hat{v}_+$ to approach their asymptotic values; the effect of the transient phase is still visible, for $t \geqslant t_{\rm r}$, in the error term $\mathcal{O}(e^{-\alpha(t)}/\sigma^2)$. The metastable time scale $t_{\rm K} = e^{R^2/2\sigma^2}$, often called Kramers' time, measures roughly the time needed for a substantial fraction of paths to reach the unstable orbit. There is no sharp transition, however, between metastable and asymptotic regime, one rather expects the density

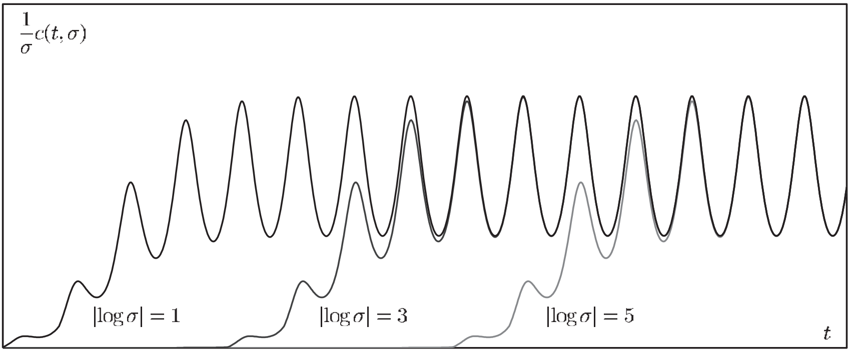

Fig. 3. The prefactor $c(t, \sigma)$ of the first-passage density as function of time, for different values of σ. After a transient regime of duration $2\,|\log \sigma|/\lambda$, the density approaches a periodic function of time. Only for times exponentially large in $1/\sigma^2$ will the density show a visible decay.

$p_+(t)$ to decrease geometrically from one period to the next one for all times $t \gg t_{\rm r}$.

Oscillatory Behaviour and Cycling

Consider now the behaviour of the first-passage density $p_+(t)$ during one given period $[nT, (n+1)\,T]$ in the metastable regime. The leading term in (2.23) depends periodically on t and $|\log \sigma|$. For fixed noise intensity σ, $p_+(t)$ is close to a periodic function of t, see Fig. 3. More surprisingly, for fixed t (which also means a fixed position on the unstable orbit), p_+ is

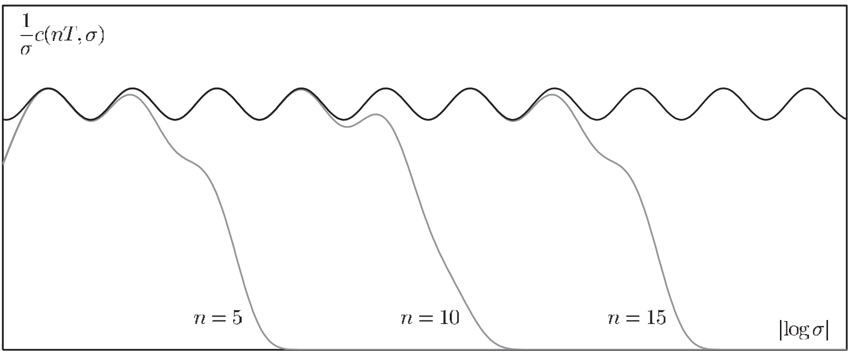

Fig. 4. The prefactor of the first-passage density as function of $|\log \sigma|$, for three different values of $t = nT$. The prefactor is close to a periodic function of $|\log \sigma|$ for $2\,|\log \sigma| \leqslant \alpha(nT) = \lambda nT$, and decreases exponentially, with an exponent proportional to $-\text{const}\,\mathrm{e}^{-\alpha(nT)}/\sigma^2$, for $2\,|\log \sigma| > \alpha(nT) = \lambda nT$.

proportional to a term depending periodically on $|\log\sigma|$ (Fig. 4)—this is the phenomenon pointed out in ref. 21.

It is, however, fundamental to consider the joint dependence of p_+ on t and σ: The leading term in (2.23) can be viewed as a periodic "profile" function of $|\log\sigma|-\theta(t)$, modulated by the periodic function $\theta'(t)$. The maximum of $t\mapsto p_+(t)$ moves once around the unstable orbit as $|\log\sigma|$ increases by λT: We thus recover the phenomenon of *cycling* described in refs. 10 and 11 for the first-exit location, but in addition we obtain here the same behaviour for the first-passage time, which keeps track of the winding number around the unstable orbit. Also note that the probability of reaching the unstable orbit during $[nT,(n+1)T]$ is proportional to

$$\int_{nT}^{(n+1)T}\theta'(t)\,P\left(\frac{|\log\sigma|-\theta(t)}{\lambda T}\right)\mathrm{d}t=\lambda T\int_{x_0-1}^{x_0}P(x)\,\mathrm{d}x=\frac{1}{2}, \qquad (2.35)$$

where we have set $x_0=(|\log\sigma|-\theta(nT))/\lambda T$. While the peak of $p_+(t)$ moves around the unstable orbit as σ decreases, changing its height periodically, the area below $p_+(t)$ remains constant. Another way to interpret (2.23) is to consider $\theta(t)$ as intrinsic time: the first-passage density expressed with respect to the time $\theta(t)$ is translated around the unstable orbit as σ decreases, with constant "velocity," without changing its shape.

Bottleneck Effect

Paths reach the intermediate level $1-\delta_1$ at times concentrated in small windows around $s^\star+kT$. When approaching the unstable orbit, they are strongly dispersed. As a result, a path reaching $y=+1$ at a fixed time $t\in[nT,(n+1)T)$ may have idled along the unstable orbit for an almost arbitrarily long time span. The probability that it has come through the kth window is proportional to (2.33). When translating this from time to space, all windows become superimposed in $\varphi^\star=(2\pi/T)\,s^\star$, meaning that with high probability, all paths reaching the unstable orbit have crossed the curve $y=1-\delta_1$ through the same small bottleneck. In fact, the same is true for the crossing of any curve bounded away from the periodic orbits. We thus recover the well-known fact that the transition between the orbits is likely to occur in a small neighbourhood of a fixed trajectory spiraling away from the stable orbit, the so-called most probable exit path. But note that the concept of most probable exit paths becomes irrelevant for the dynamics close to the unstable orbit.

High-Frequency Limit $T\ll 1$

We now examine how the first-passage density changes as a function of the period T. By this we mean that the functions a and g are scaled by a factor T, that is, $a(t)=a_T(t)=a_1(t/T)$ and $g(t)=g_T(t)=g_1(t/T)$ for some fixed functions a_1 and g_1. In particular, $\alpha(t) = T\alpha_1(t/T)$, so that, for instance, (2.14) becomes

$$v_-^{\text{per}}(Ts) = \frac{T}{1-\mathrm{e}^{-2\lambda T}} \int_s^{s+1} \mathrm{e}^{-2T\alpha_1(s+1,\,u)} g_1(u)^2 \,\mathrm{d}u. \tag{2.36}$$

In the limit $T \to 0$, the functions v_-^{per} and $\hat{v}_+^{\text{per}}$ both approach a constant value $\langle g_1^2 \rangle / 2\lambda$, where $\langle g_1^2 \rangle$ denotes the average value of g_1^2. The exponential rate R^2 approaches the value $2\lambda[\delta_1^2 + (2-\delta_1)^2]/\langle g_1^2 \rangle$ (of course, we cannot actually take the limit $T \to 0$, because ρ^{per} would also become constant in this limit, so that the discussion is to be understood for small but finite T).

Since $|\Gamma(1-\mathrm{i}y)|$ decreases exponentially with $|y|$, $P(x)$ is close to a sinusoïd with amplitude of order $\mathrm{e}^{-\text{const}/T}$, so that the cycling profile becomes flat in the high-frequency limit, cf. Fig. 5. This is related to the fact that many terms, i.e., paths with many different winding numbers, contribute to the sum (2.28). In the limit, the cycling velocity $1/\theta'(t)$ behaves like $\langle g_1^2 \rangle / \lambda g(t)^2$, so that the first-passage density $p_+(t)$ behaves like the noise coefficient $g(t)^2$, independently of $a(t)$.

Adiabatic Limit $T \gg 1$

In the low-frequency limit, $v_-^{\text{per}}(t)$ and $\hat{v}_+^{\text{per}}(t)$ both follow adiabatically $v^\star(t)$, at a distance of order $1/T$. As a result, the exponential rate R^2 approaches $[\delta_1^2 + (2-\delta_1)^2]/\bar{v}$, which is smaller than its value in the limit $T \to 0$ (because $2\lambda\bar{v} > \langle 2a_1 v^\star \rangle = \langle g_1^2 \rangle$). This is due to the fact that paths have enough time to probe for the moment when the transition costs the least.

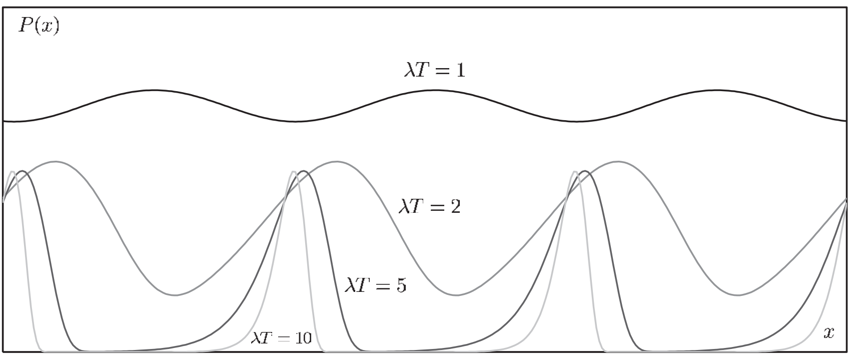

Fig. 5. The cycling profile $P(x)$, plotted over three periods, for four different values of λT. The normalization is such that the integral of $P(x)$ over one period is equal to $1/2\lambda T$.

The sum (2.28) is dominated by one term, so that the cycling profile $P(x)$ is sharply peaked, cf. Fig. 5. The first-passage density is thus dominated by paths making a fixed number of revolutions around the unstable orbit. The cycling velocity $1/\theta'(t)$ converges to $1/a(t)$. The first-passage density $p_+(t)$ thus consists of a peak of height proportional to $a(t)$, moving around with velocity $1/a(t)$.

In spite of the fact that $P(x)$ is sharply peaked, one can show that the first-passage density at fixed t is still a monotonously decreasing function of σ. This is due to the exponentially small factor $\mathrm{e}^{-R^2/2\sigma^2}$.

3. THE RENEWAL EQUATION

In this section, we establish a renewal equation satisfied by the first-passage time τ_+ at $+1$ of the process y_t defined in Section 2.2. By restarting the process at the first time $\tau_\nearrow = \tau_\nearrow(0,-1)$ at which the level $1-\delta_1$ is reached, the distribution function of τ_+ can be written as

$$\begin{aligned}\mathbb{P}^{0,-1}\{\tau_+ \leqslant t\} &= \mathbb{E}^{0,-1}\{1_{\{\tau_\nearrow \leqslant t\}} \mathbb{P}^{\tau_\nearrow, 1-\delta_1}\{\tau_+ \leqslant t\}\} \\ &= \int_0^t Q(t,s)\,\psi_-(s,0)\,\mathrm{d}s, \end{aligned} \tag{3.1}$$

where we have introduced the quantities

$$Q(t,s) = \mathbb{P}^{s,1-\delta_1}\{\tau_+ \leqslant t\} \tag{3.2}$$

$$\psi_-(s,0) = \frac{\partial}{\partial s}\mathbb{P}^{0,-1}\{\tau_\nearrow \leqslant s\}. \tag{3.3}$$

The first-passage density $\psi_-(s,0)$ of y_s to $1-\delta_1$ depends only on the linear process y_s^-; we will discuss its computation in Section 4.1. The function $Q(t,s)$ depends on all subsequent switchings of the process y_t between y_t^+ and y_t^-, and therefore we will write it as the solution of an integral equation, that will serve as a *renewal equation*. It is obtained by restarting the process each time the level $1-\delta_1$ is reached from below.

Let $y_t^\#$ denote the stochastic process obtained by killing y_t^+ upon first hitting the level $+1$, and introduce the stopping time

$$\tau_\searrow^\#(t_0) = \inf\{s > t_0 : y_s^\# < 1-\delta_2\}, \tag{3.4}$$

with the convention that $\tau_\searrow^\# = \infty$ if y_s^+ either hits level $+1$ before reaching $1-\delta_2$ or never reaches $1-\delta_2$.

Proposition 3.1. $Q(t,s)$ satisfies the *renewal equation*

$$Q(t,s) = P_1(t,s) + \int_s^t Q(t,u)\, K(u,s)\, \mathrm{d}u, \tag{3.5}$$

where

$$P_1(t,s) = \mathbb{P}^{s,1-\delta_1}\{\tau_+ \leqslant t, \tau_+ < \tau_\searrow\}, \tag{3.6}$$

$$K(u,s) = \int_s^u \psi_\nearrow(u,v)\, \psi_\searrow^\#(v,s)\, \mathrm{d}v, \tag{3.7}$$

$$\psi_\nearrow(u,v) = \frac{\partial}{\partial u} \mathbb{P}^{v,1-\delta_2}\{\tau_\nearrow \leqslant u\}, \tag{3.8}$$

$$\psi_\searrow^\#(v,s) = \frac{\partial}{\partial v} \mathbb{P}^{s,1-\delta_1}\{\tau_\searrow^\# \leqslant v\}. \tag{3.9}$$

Proof. Splitting up the event according to whether $\tau_\searrow > \tau_+$ or $\tau_\searrow \leqslant \tau_+$, we can write

$$Q(t,s) = P_1(t,s) + \mathbb{P}^{s,1-\delta_1}\{\tau_\searrow \leqslant \tau_+ \leqslant t\}. \tag{3.10}$$

Note that $\tau_\searrow \leqslant \tau_+ \leqslant t$ if and only if the killed process $y_t^\#$ first reaches $1-\delta_2$ and then $+1$, and both events occur before time t. Therefore, the second term on the right-hand side of (3.10) can be written as

$$\begin{aligned}
&\mathbb{E}^{s,1-\delta_1}\{1_{\{\tau_\searrow^\# \leqslant t\}} \mathbb{P}^{\tau_\searrow^\#,1-\delta_2}\{\tau_+ \leqslant t\}\} \\
&= \mathbb{E}^{s,1-\delta_1}\{1_{\{\tau_\searrow^\# \leqslant t\}} \mathbb{E}^{\tau_\searrow^\#,1-\delta_2}\{1_{\{\tau_\nearrow \leqslant t\}} Q(t,\tau_\nearrow)\}\} \\
&= \int_s^t \int_v^t Q(t,u)\, \psi_\nearrow(u,v)\, \mathrm{d}u\, \psi_\searrow^\#(v,s)\, \mathrm{d}v.
\end{aligned} \tag{3.11}$$

Using Fubini's theorem to interchange the integrals, we obtain the second term in (3.5). ∎

Note that since $Q(t,t)=0$, the derivatives

$$q(t,s) = \frac{\partial}{\partial t} Q(t,s), \qquad p_1(t,s) = \frac{\partial}{\partial t} P_1(t,s) \tag{3.12}$$

satisfy the similar renewal equation

$$q(t,s) = p_1(t,s) + \int_s^t q(t,u)\, K(u,s)\, \mathrm{d}u. \tag{3.13}$$

The renewal equation (3.5) can be solved by iterations. In our case, K will be so small that only the first term needs to be computed. The convergence of the iterative method can, however, easily be controlled.

Proposition 3.2. Let $K_1(u,s) = K(u,s)$ and define for $n \in \mathbb{N}$

$$K_{n+1}(u,s) = \int_s^u K(u,v)\, K_n(v,s)\, \mathrm{d}v. \tag{3.14}$$

Then, for any $N \geqslant 1$, the solution of the renewal equation (3.5) satisfies

$$Q(t,s) = P_1(t,s) + \sum_{n=1}^{N} \int_s^t P_1(t,u)\, K_n(u,s)\, \mathrm{d}u + R_{N+1}(t,s), \tag{3.15}$$

with a remainder satisfying $0 \leqslant R_N(t,s) \leqslant M^N\, |t-s|^N / N!$ for some constant M which depends neither on N nor on t, s. Hence, $Q(t,s)$ can be written as a series which converges uniformly on compact sets.

Proof. First note that $\psi_{\nearrow}(t,u)$ is bounded by some constant $M > 0$ which implies that $K(t,s) \leqslant M$ for all t, s. By induction, we see that (3.15) holds with

$$R_N(t,s) = \int_s^t Q(t,v)\, K_N(v,s)\, \mathrm{d}v \tag{3.16}$$

and that

$$\int_s^t K_N(v,s)\, \mathrm{d}v \leqslant M^N\, |t-s|^N / N! \tag{3.17}$$

Thus the bounds on $R_N(t,s)$ follow from the trivial bounds $0 \leqslant Q(t,v) \leqslant 1$. ∎

The nth term in the sum in (3.15) is the contribution of those paths which cross the levels $1-\delta_1$ and $1-\delta_2$ alternatively n times during the time interval $[s,t]$, before finally reaching $+1$.

4. THE FIRST-PASSAGE DENSITIES

4.1. Leaving the Stable Orbit

In this section, we ignore all possible switchings between the processes y_t^+ and y_t^- in the definition of y_t and focus on the stochastic process $y_t^-(t_0, -1)$ only. Recall that this stochastic process is defined by the SDE

$$\mathrm{d}y_t^- = -a(t)(y_t^- + 1)\, \mathrm{d}t + \sigma g(t)\, \mathrm{d}W_t, \qquad y_{t_0}^- = -1, \tag{4.1}$$

for some initial time t_0. Thus it is a Gaussian process, given by

$$y_t = y_t^-(t_0, -1) = -1 + \sigma \int_{t_0}^{t} \mathrm{e}^{-\alpha(t,s)} g(s)\, \mathrm{d}W_s, \tag{4.2}$$

where $\alpha(t, s) = \alpha(t) - \alpha(s) = \int_s^t a(u)\, \mathrm{d}u$. At time t, the Gaussian random variable y_t^- has mean -1 and variance $\sigma^2 v_-(t, t_0)$, where

$$v_-(t, t_0) = \int_{t_0}^{t} \mathrm{e}^{-2\alpha(t,s)} g(s)^2\, \mathrm{d}s. \tag{4.3}$$

Note that $v_-(t, t_0)$ satisfies the deterministic differential equation

$$\frac{\partial}{\partial t} v_-(t, t_0) = -2a(t)\, v_-(t, t_0) + g(t)^2, \tag{4.4}$$

with initial condition $v_-(t_0, t_0) = 0$. This equation also admits a periodic solution

$$v_-^{\mathrm{per}}(t) = \frac{1}{1 - \mathrm{e}^{-2\lambda T}} \int_t^{t+T} \mathrm{e}^{-2\alpha(t+T,s)} g(s)^2\, \mathrm{d}s. \tag{4.5}$$

Recall that $\lambda = \alpha(T)/T$ is the Lyapunov exponent. These two solutions of (4.4) are related by

$$v_-(t, t_0) = v_-^{\mathrm{per}}(t) - \mathrm{e}^{-2\alpha(t,t_0)} v_-^{\mathrm{per}}(t_0), \tag{4.6}$$

see Fig. 6.

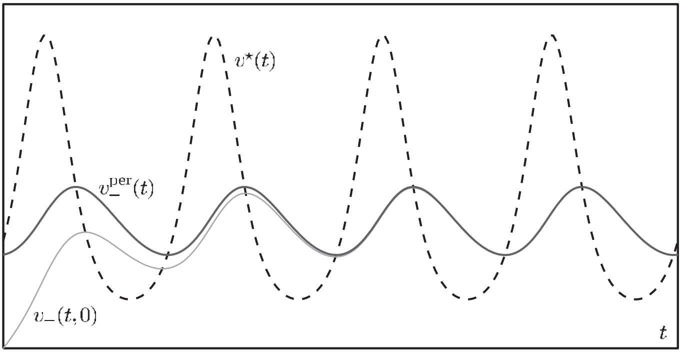

Fig. 6. The functions $v_-(t, 0)$ and $v_-^{\mathrm{per}}(t)$.

The following result describes the behaviour of the first-passage density $\psi_-(t,0)$ of $y_t^-(0,-1)$ to $1-\delta_1$.

Proposition 4.1. Assume that $v_-^{\mathrm{per}}(t) \leqslant 2v^\star(t)(1-\Delta)$ for some constant $\Delta > 0$. Then the first-passage density $\psi_-(t,0)$ can be written as

$$\psi_-(t,0) = \frac{1}{\sigma} c_-(t,0)\, \mathrm{e}^{-\rho_-(t,0)^2/2\sigma^2}, \tag{4.7}$$

where

$$\rho_-(t,0)^2 = \frac{(2-\delta_1)^2}{v_-(t,0)} \tag{4.8}$$

and

$$c_-(t,0) = \frac{2-\delta_1}{\sqrt{2\pi}} \left[\frac{1}{v_-(t,0)} - \frac{1}{2v^\star(t)}\right] \frac{g(t)^2}{\sqrt{v_-(t,0)}} \left[1 + \frac{1}{\Delta}\mathcal{O}\left(\frac{\sigma}{\Delta^2} + \frac{\mathrm{e}^{-\mathrm{const}\,\Delta^2/\sigma^2}}{\sigma} t\right)\right] \tag{4.9}$$

for all $t < \frac{1}{2}\sigma(1-\sigma/\Delta^2)\,\mathrm{e}^{\mathrm{const}\,\Delta^2/\sigma^2}$.

Proof. The process $z_t^- = \mathrm{e}^{\alpha(t)}(y_t^- + 1)$ satisfies a stochastic differential equation without drift term,

$$\mathrm{d}z_t^- = \sigma \mathrm{e}^{\alpha(t)} g(t)\,\mathrm{d}W_t, \qquad z_0^- = 0. \tag{4.10}$$

It is a Gaussian process, with mean zero and variance $\sigma^2\tilde{v}_-(t,0) = \sigma^2\mathrm{e}^{2\alpha(t)}v_-(t,0)$. The first passage of y_t^- at $1-\delta_1$ corresponds to the first passage of z_t^- at the time-dependent level $d_-(t) = (2-\delta_1)\,\mathrm{e}^{\alpha(t)}$. Some properties of first-passage densities of such Gaussian processes are discussed in the appendix.

We will apply Corollary A.5 from the appendix, after checking that $\tilde{v}_-(t,0)$ and $d_-(t)$ satisfy the required conditions. (A.11) can easily be verified by rewriting the condition with the help of $v_-^{\mathrm{per}}(t)$ and recalling the assumption on the relation between $v_-^{\mathrm{per}}(t)$ and $v^\star(t)$. Note that Δ in this assumption and Δ in (A.11) differ by a constant. Checking (A.13) is even easier, just keep in mind that $v_-(t,0)$ approaches $v_-^{\mathrm{per}}(t)$ and thus is bounded. Next, (A.12) can be established by showing first that there exists a $\delta_0 > 0$ such that $|\tilde{c}(t,s)|$ as defined in the appendix is bounded above by const $\sqrt{t-s}$ as long as $t-s \leqslant \delta_0$. For $t-s > \delta_0$, one can show that $|\tilde{c}(t,s)|$ is at most of order one. Finally, Assumption (A.22) is seen to be satisfied for large enough M_3 by comparing again $v_-(t,0)$ with $v_-^{\mathrm{per}}(t)$. Applying the corollary immediately yields (4.7)–(4.9). ∎

4.2. Reaching the Unstable Orbit

We now turn to the process $y_t^+(s, 1-\delta_1)$, defined by the SDE

$$\mathrm{d}y_t^+ = a(t)(y_t^+ - 1)\,\mathrm{d}t + \sigma g(t)\,\mathrm{d}W_t, \qquad y_s^+ = 1-\delta_1, \tag{4.11}$$

for some initial time s. It is given by

$$y_t^+ = y_t^+(s, 1-\delta_1) = 1 - \delta_1 \mathrm{e}^{\alpha(t,s)} + \sigma \int_s^t \mathrm{e}^{\alpha(t,u)} g(u)\,\mathrm{d}W_u. \tag{4.12}$$

At time t, y_t^+ is Gaussian with variance $\sigma^2 v_+(t,s)$, where

$$v_+(t,s) = \int_s^t \mathrm{e}^{2\alpha(t,u)} g(u)^2\,\mathrm{d}u. \tag{4.13}$$

As y_t^+ is spreading fast as t increases, it is helpful to consider the stochastic process

$$z_t^+ = \delta_1 + \mathrm{e}^{-\alpha(t,s)}[y_t^+ - 1], \tag{4.14}$$

which is the solution of the SDE

$$\mathrm{d}z_t^+ = \sigma \mathrm{e}^{-\alpha(t,s)} g(t)\,\mathrm{d}W_t, \qquad z_s^+ = 0. \tag{4.15}$$

It is also Gaussian and has variance $\sigma^2 \hat{v}_+(t,s)$, where

$$\hat{v}_+(t,s) = \mathrm{e}^{-2\alpha(t,s)} v_+(t,s) = \int_s^t \mathrm{e}^{-2\alpha(u,s)} g(u)^2\,\mathrm{d}u. \tag{4.16}$$

Note that $s \mapsto \hat{v}_+(t,s)$ satisfies the deterministic differential equation

$$\frac{\partial}{\partial s}\hat{v}_+(t,s) = 2a(s)\,\hat{v}_+(t,s) - g(s)^2, \tag{4.17}$$

with $\hat{v}_+(t,t) = 0$. This equation also admits a periodic solution

$$\hat{v}_+^{\mathrm{per}}(s) = \frac{1}{\mathrm{e}^{2\lambda T} - 1} \int_s^{s+T} \mathrm{e}^{2\alpha(s+T,u)} g(u)^2\,\mathrm{d}u. \tag{4.18}$$

The function $\hat{v}_+(t,s)$ can then be determined by the relation

$$\hat{v}_+(t,s) = \hat{v}_+^{\mathrm{per}}(s) - \mathrm{e}^{-2\alpha(t,s)} \hat{v}_+^{\mathrm{per}}(t), \tag{4.19}$$

see Fig. 6. By the reflection principle, the distribution function of the first-passage time $\tilde{\tau}_+$ of y_t^+ at $+1$ is given by

$$\mathbb{P}^{s,\,1-\delta_1}\{\tilde{\tau}_+ \leqslant t\} = 2\mathbb{P}^{s,\,1-\delta_1}\{y_t^+ \geqslant 1\} = 2\Phi\left(-\frac{\rho_+(t,s)}{\sigma}\right), \tag{4.20}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}u$ denotes the distribution function of the standard normal law, and

$$\rho_+(t,s)^2 = \frac{\delta_1^2 \mathrm{e}^{2\alpha(t,s)}}{v_+(t,s)} = \frac{\delta_1^2}{\hat{v}_+(t,s)}. \tag{4.21}$$

The density of $\tilde{\tau}_+$ can thus be written as

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{P}^{s,\,1-\delta_1}\{\tilde{\tau}_+ \leqslant t\} &= \frac{1}{\sigma}\frac{\delta_1}{\sqrt{2\pi}}\frac{\partial_t \hat{v}_+(t,s)}{\hat{v}_+(t,s)^{3/2}}\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2}\\ &= \frac{1}{\sigma}\frac{\delta_1}{\sqrt{2\pi}}\frac{g(t)^2\mathrm{e}^{-2\alpha(t,s)}}{\hat{v}_+(t,s)^{3/2}}\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2}.\end{aligned} \tag{4.22}$$

Let us now show that the function

$$p_1(t,s) = \frac{\partial}{\partial t}\mathbb{P}^{s,\,1-\delta_1}\{\tau_+ \leqslant t,\, \tau_+ < \tau_{\searrow}\} = \frac{\partial}{\partial t}\mathbb{P}^{s,\,1-\delta_1}\{\tilde{\tau}_+ \leqslant t,\, \tilde{\tau}_+ < \tau_{\searrow}\} \tag{4.23}$$

behaves in the same way. Recall that $\tau_{\searrow} = \tau_{\searrow}(s, 1-\delta_1)$ denotes the first-passage time of $y_t^+(s, 1-\delta_1)$ at the lower level $1-\delta_2$. It is obvious that $p_1(t,s)$ is bounded above by (4.22), but we want to show that it actually has the same exponential asymptotics, and, moreover, almost the same prefactor. In other words, it is not only unlikely that a path reaches the unstable orbit, but also the conditional probability that a path returns to $1-\delta_2$ before reaching $+1$, given that it actually reaches the unstable orbit at $+1$, is small. Hypothesis H3 plays a crucial role here.

In a first step, we study the density of $\tau_{\searrow}$.

Lemma 4.2. The density $\psi_{\searrow}(u,s) = \frac{\partial}{\partial u}\mathbb{P}^{s,\,1-\delta_1}\{\tau_{\searrow} \leqslant u\}$ of $\tau_{\searrow} = \tau_{\searrow}(s, 1-\delta_1)$ satisfies

$$\psi_{\searrow}(u,s) = \frac{1}{\sigma}c_{\searrow}(u,s)\,\mathrm{e}^{-\rho_{\searrow}(u,s)^2/2\sigma^2}, \tag{4.24}$$

where the prefactor $c_{\searrow}(u,s)$ is bounded by a constant times $\hat{v}_+(u,s)^{-3/2}\,\mathrm{e}^{-\alpha(u,s)}$ and

$$\rho_{\searrow}(u,s)^2 = \frac{[\delta_1 - \delta_2\mathrm{e}^{-\alpha(u,s)}]^2}{\hat{v}_+(u,s)}. \tag{4.25}$$

Proof. Recall the definition of the stochastic process $z_u^+ = \delta_1 + \mathrm{e}^{-\alpha(u,s)}[y_u^+ - 1]$ from (4.14). The first passage of y_u^+ at $1-\delta_2$ coincides with the first passage of z_u^+ at the time-dependent level $d_+(u) = \delta_1 - \delta_2 \mathrm{e}^{-\alpha(u,s)}$. Therefore, the result follows from Lemma A.1 of the appendix which we apply with $d(u) = d_+(u)$ and $v(u) = \hat{v}_+(u,s)$ for fixed s. ∎

Proposition 4.3. Assume that $\hat{v}_+^{\mathrm{per}}(u) \leqslant 2v^\star(u)(1-\Delta)$ for all u. Then

$$p_1(t,s) = \frac{1}{\sigma}\, c_+(t,s)\, \mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2}, \tag{4.26}$$

where $\rho_+(t,s)$ is the function defined in (4.21) and

$$c_+(t,s) = \frac{\delta_1}{\sqrt{2\pi}} \frac{g(t)^2 \mathrm{e}^{-2\alpha(t,s)}}{\hat{v}_+(t,s)^{3/2}} - \frac{1}{\sigma}\, \mathcal{O}(\mathrm{e}^{-\alpha(t,s)} \mathrm{e}^{-\mathrm{const}\, \Delta_0^2/\sigma^2}), \tag{4.27}$$

where $\Delta_0 = \Delta/(1-\Delta) \wedge (\delta_2 - \delta_1)/\delta_1 \wedge 1$. If the condition on $\hat{v}_+^{\mathrm{per}}$ is not satisfied, (4.26) remains valid, but the prefactor can be smaller than (4.27).

Proof. The main idea of the proof is that the condition on $\hat{v}_+^{\mathrm{per}}$ excludes that the most probable path going from $(s, 1-\delta_1)$ to $(t,1)$ crosses the lower level $1-\delta_2$. The only difficulty resides in exploiting this fact in order to obtain the exponentially small error bound in (4.27).

By definition of $P_1(t,s)$,

$$P_1(t,s) = \mathbb{P}^{s,1-\delta_1}\{\tilde{\tau}_+ \leqslant t\} - \mathbb{P}^{s,1-\delta_1}\{\tau_\searrow < \tilde{\tau}_+ \leqslant t\}. \tag{4.28}$$

Recall that $\tau_\searrow^\#$ denotes the first-passage time at level $1-\delta_2$ for the process $y_t^\#$, which is obtained by killing y_t^+ upon reaching level $+1$. The second term on the right-hand side can be written as

$$\begin{aligned}\mathbb{P}^{s,1-\delta_1}\{\tau_\searrow < \tilde{\tau}_+ \leqslant t\} &= \mathbb{E}^{s,1-\delta_1}\{1_{\{\tau_\searrow^\# \leqslant t\}} \mathbb{P}^{\tau_\searrow^\#, 1-\delta_2}\{\tilde{\tau}_+ \leqslant t\}\} \\ &= \int_s^t \mathbb{P}^{u,1-\delta_2}\{\tilde{\tau}_+ \leqslant t\}\, \psi_\searrow^\#(u,s)\, \mathrm{d}u. \end{aligned} \tag{4.29}$$

Since $\mathbb{P}^{t,1-\delta_2}\{\tilde{\tau}_+ \leqslant t\} = 0$ and $\psi_\searrow^\#(u,s) \leqslant \psi_\searrow(u,s)$, we have

$$\frac{\partial}{\partial t} \mathbb{P}^{s,1-\delta_1}\{\tau_\searrow < \tilde{\tau}_+ \leqslant t\} \leqslant \int_s^t \frac{\partial}{\partial t} \mathbb{P}^{u,1-\delta_2}\{\tilde{\tau}_+ \leqslant t\}\, \psi_\searrow(u,s)\, \mathrm{d}u. \tag{4.30}$$

The first term in the integrand is similar to (4.22), the only difference lying in the initial condition. We can write it in the form

$$\frac{\partial}{\partial t} \mathbb{P}^{u,1-\delta_2}\{\tilde{\tau}_+ \leqslant t\} = \frac{1}{\sigma}\, \tilde{c}_+(t,u)\, \mathrm{e}^{-\tilde{\rho}_+(t,u)^2/2\sigma^2}, \tag{4.31}$$

where $\tilde{\rho}_+(t,u)^2 = \delta_2^2/\hat{v}_+(t,u)$ and $\tilde{c}_+(t,u) = \delta_2\, g(t)^2\, \mathrm{e}^{-2\alpha(t,u)}/\sqrt{2\pi}\,\hat{v}_+(t,u)^{3/2}$. The derivative $p_1(t,s)$ thus satisfies

$$\begin{aligned} p_1(t,s) \geqslant{}& \frac{1}{\sigma}\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2} \\ &\times\left[\frac{\delta_1}{\sqrt{2\pi}}\frac{g(t)^2\,\mathrm{e}^{-2\alpha(t,s)}}{\hat{v}_+(t,s)^{3/2}} - \frac{1}{\sigma}\int_s^t \tilde{c}_+(t,u)\, c_\searrow(u,s)\,\mathrm{e}^{-\chi(u)\,\rho_+(t,s)^2/2\sigma^2}\,\mathrm{d}u\right], \end{aligned} \tag{4.32}$$

where

$$\chi(u) = \frac{\tilde{\rho}_+(t,u)^2 + \rho_\searrow(u,s)^2}{\rho_+(t,s)^2} - 1. \tag{4.33}$$

Assume that $t-s>2$. (For $t-s\leqslant 2$, it suffices to apply the first one of the arguments below.) We split the integral in (4.32) at $u_1 = s+1$ and $u_2 = t-1$. The integrals over $[s,u_1]$ and $[u_2,t]$ are easily seen to be exponentially small because $\chi(u)$ diverges as $u\to s$ and $u\to t$ (in fact, for small enough σ, the integrand is maximal at u_1 or u_2, respectively). In order to bound the remaining integral over $[u_1,u_2]$, we only need to find a positive lower bound for the function $\chi(u)$, valid whenever $s\leqslant u\leqslant t$. Using the fact that $\hat{v}_+(t,u) = \mathrm{e}^{2\alpha(u,s)}[\hat{v}_+(t,s) - \hat{v}_+(u,s)]$, it is straightforward to show that

$$\chi(u) = \frac{[\kappa(u) - (1-r(u))]^2}{r(u)(1-r(u))}, \tag{4.34}$$

where

$$\kappa(u) = \frac{\delta_2}{\delta_1}\,\mathrm{e}^{-\alpha(u,s)} \qquad \text{and} \qquad r(u) = \frac{\hat{v}_+(u,s)}{\hat{v}_+(t,s)} \in [0,1]. \tag{4.35}$$

We may assume the existence of a constant $c^\star > 1$ such that $\hat{v}_+(t,u)/\hat{v}_+(t,s)\leqslant c^\star$ for all $u\in[s,t]$. The function $r(u)$ being increasing with range $[0,1]$, we can define $u^\star\in(s,t)$ by

$$1 - r(u^\star) = \mathrm{e}^{-2\alpha(u^\star,s)}\,\frac{\hat{v}_+(t,u^\star)}{\hat{v}_+(t,s)} = \frac{1}{c^\star}. \tag{4.36}$$

Consider first the case $u\geqslant u^\star$. Then

$$\frac{\kappa(u)^2}{1-r(u)} = \left(\frac{\delta_2}{\delta_1}\right)^2 \frac{\hat{v}_+(t,s)}{\hat{v}_+(t,u)} \geqslant \left(\frac{\delta_2}{\delta_1}\right)^2 \frac{1}{c^\star} \geqslant \left(\frac{\delta_2}{\delta_1}\right)^2 (1-r(u)), \tag{4.37}$$

and thus

$$\chi(u) \geqslant \frac{[\kappa(u)-(1-r(u))]^2}{1-r(u)} \geqslant \frac{1}{c^\star}\left(\frac{\delta_2}{\delta_1}-1\right)^2 = \frac{(\delta_2-\delta_1)^2}{c^\star\delta_1^2}. \tag{4.38}$$

We turn now to the case $u \leqslant u^\star$, in which it suffices to find a lower bound for $\theta(u) = \kappa(u)+r(u)-1$, since $\chi(u) \geqslant 4\theta(u)^2$ for all u. First note that $\theta(s) = (\delta_2-\delta_1)/\delta_1$ and, since (4.36) implies $\mathrm{e}^{-\alpha(u^\star, s)} \geqslant 1/c^\star$, we also know that $\theta(u^\star) \geqslant (\delta_2-\delta_1)/c^\star\delta_1$. Thus if $\theta(u)$ reaches its minimum on the boundary of $[s, u^\star]$, we are done. Otherwise, the fact that

$$\theta'(u) = 0 \Leftrightarrow \kappa(u) = 2v^\star(u)\,\frac{\mathrm{e}^{-2\alpha(u, s)}}{\hat{v}_+(t, s)} \tag{4.39}$$

shows that if $\theta(u)$ reaches its minimum in (s, t), then

$$\theta(u) = \frac{\mathrm{e}^{-2\alpha(u, s)}}{\hat{v}_+(t, s)}\,[2v^\star(u)-\hat{v}_+(t, u)] \geqslant \frac{1}{c^\star}\left[\frac{2v^\star(u)}{\hat{v}_+(t, u)}-1\right] \geqslant \frac{\Delta}{c^\star(1-\Delta)}, \tag{4.40}$$

where the last estimate holds due to our assumption on $\hat{v}_+^{\mathrm{per}}(u)$. Note that if $\hat{v}_+^{\mathrm{per}}(u)$ is not bounded away from $2v^\star(u)$, then the minimum of θ is still positive, but may be of the order $\mathrm{e}^{-2\alpha(t, s)}$, which can become very small. ∎

4.3. The Renewal Kernel

The kernel $K(u, s)$ involves the functions $\psi_{\nearrow}(u, v)$ and $\psi^{\#}_{\searrow}(v, s)$ defined in (3.8) and (3.9). We already know from Lemma 4.2 that the rate associated with $\psi_{\searrow}(v, s)$, which provides an upper bound for $\psi^{\#}_{\searrow}(v, s)$, is given by

$$\rho_{\searrow}(v, s)^2 = \frac{[\delta_1-\delta_2\mathrm{e}^{-\alpha(v, s)}]^2}{\hat{v}_+(v, s)}. \tag{4.41}$$

Note that $\rho_{\searrow}(v, s)$ vanishes at a time $v \geqslant s$ such that $\mathrm{e}^{-\alpha(v, s)} = \delta_1/\delta_2$, which shows that most paths starting at level $1-\delta_1$ at time s will reach the lower level $1-\delta_2$ close to that time.

The same argument as in Proposition 4.1, applied for a different initial condition, shows that $\psi_{\nearrow}(u, v)$ can be written as $\sigma^{-1}c_{\nearrow}(u, v)\,\mathrm{e}^{-\rho_{\nearrow}(u, v)^2/2\sigma^2}$, with a rate

$$\rho_{\nearrow}(u, v)^2 = \frac{[(2-\delta_1)-(2-\delta_2)\,\mathrm{e}^{-\alpha(u, v)}]^2}{v_-(u, v)}, \tag{4.42}$$

which is bounded below by $(\delta_2-\delta_1)^2/\bar{v}$. It thus follows from the Definition (3.7) of K that $K(u,s) \leqslant (\text{const}/\sigma)\,\mathrm{e}^{-(\delta_2-\delta_1)^2/2\bar{v}\sigma^2}$.

By differentiating (3.15), we find

$$q(t,s) = p_1(t,s) + \sum_{n=1}^{\infty} \int_s^t p_1(t,u)\, K_n(u,s)\, \mathrm{d}u. \tag{4.43}$$

(Note that the sum is converging uniformly on compact sets.) The smallness of K is not yet sufficient to ensure the smallness of the sum, relatively to $p_1(t,s)$. The following result provides a sufficient bound.

Proposition 4.4. Let $\bar{p}_1(t,s) := \frac{1}{\sigma}(c_+(t,s) \vee 1)\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2} \geqslant p_1(t,s)$.[4] Assume that $\hat{v}_+^{\text{per}}(u) \leqslant 2v^\star(u)(1-\Delta)$ for all u. Then, the relation

$$\int_s^t \bar{p}_1(t,u)\, K_n(u,s)\, \mathrm{d}u \leqslant \left[\text{const}(t-s)\, \frac{1}{\sigma^2}\, \mathrm{e}^{-\text{const}\, \Delta_0^2/\sigma^2} \right]^n \bar{p}_1(t,s) \tag{4.44}$$

holds for all $n \geqslant 1$, with $\Delta_0 = \Delta/(1-\Delta) \wedge (\delta_2-\delta_1)/\delta_1 \wedge 1$.

Proof. The main idea consists in comparing $\psi_{\nearrow}(u,v)$ to the density $\psi_{\nearrow}^+(u,v)$ of the first-passage time of $y_t^+(v,1-\delta_2)$ at $1-\delta_1$ (recall that $\psi_{\nearrow}(u,v)$ relates to the process $y_t^-(v,1-\delta_2)$). The exponential rate $\rho_{\nearrow}^+(u,v)$ associated with $\psi_{\nearrow}^+(u,v)$ is given by

$$\rho_{\nearrow}^+(u,v)^2 = \frac{[\delta_2 - \delta_1 \mathrm{e}^{-\alpha(u,v)}]^2}{\hat{v}_+(u,v)}. \tag{4.45}$$

We claim that

$$\rho_{\nearrow}(u,v)^2 \geqslant \rho_{\nearrow}^+(u,v)^2 \tag{4.46}$$

for all $u \geqslant v$, as a consequence of Hypothesis H4. Denote $\mathrm{e}^{-\alpha(u,v)}$ by ξ and $\underline{v}/\bar{v}$ by κ^2. On the one hand, it follows from the Definitions (4.3) and (4.16) of v_- and $\hat{v}_+$ that

$$v_-(u,v) \leqslant \int_v^u g(w)^2\, \mathrm{d}w \leqslant \frac{1}{\xi^2}\, \hat{v}_+(u,v). \tag{4.47}$$

[4] We write $a \vee b$ to denote the maximum of two real numbers a and b.

On the other hand, since $g(w)^2 = 2a(w)\, v^\star(w)$ and $v^\star(w) \in [\underline{v}, \overline{v}]$, we can write

$$\frac{1}{\overline{v}}\, v_-(u,v) \leqslant \int_v^u 2a(w)\, \mathrm{e}^{-2\alpha(u,w)}\, \mathrm{d}w = 1-\xi^2 = \int_v^u 2a(w)\, \mathrm{e}^{-2\alpha(w,v)}\, \mathrm{d}w$$
$$\leqslant \frac{1}{\underline{v}}\, \hat{v}_+(u,v). \tag{4.48}$$

We can thus conclude that

$$\frac{\rho_\nearrow(u,v)}{\rho_\nearrow^+(u,v)} \geqslant \frac{(2-\delta_1)-(2-\delta_2)\,\xi}{\delta_2-\delta_1\xi}\,[\xi \vee \kappa]. \tag{4.49}$$

Using the monotonicity of the ratio in the preceding estimate, we see that relation (4.46) is satisfied provided $[(2-\delta_1)-(2-\delta_2)\,\xi]\,\xi \geqslant \delta_2-\delta_1\xi$ for all $\xi \geqslant \kappa$, which easily follows from Hypothesis H4.

In order to prove (4.44) for $n=1$, we first consider the rate $\tilde{\rho}_+(t,v)$ associated with $\mathbb{P}^{v,1-\delta_2}\{\tilde{\tau}_+ \leqslant t\}$, compare (4.31). Proceeding as in Proposition 4.3, it is straightforward to show that

$$\tilde{\rho}_+(t,v)^2 \leqslant \rho_+(t,u)^2 + \rho_\nearrow^+(u,v)^2$$
$$\leqslant \rho_+(t,u)^2 + \rho_\nearrow(u,v)^2 \tag{4.50}$$

for all $u \in [v,t]$. This is actually a consequence of the Markov property and the fact that $y_t^+(v, 1-\delta_2)$ has to cross the level $1-\delta_1$ before reaching $+1$. Equation (4.50) implies that

$$\int_s^t \overline{p}_1(t,u)\, K(u,s)\, \mathrm{d}u$$
$$\leqslant \int_s^t \int_v^t \frac{1}{\sigma^2}\,[c_+(t,u) \vee 1]\, c_\nearrow(u,v)\, \mathrm{e}^{-[\rho_+(t,u)^2+\rho_\nearrow(u,v)^2]/2\sigma^2}\, \mathrm{d}u\, \psi_\searrow(v,s)\, \mathrm{d}v$$
$$\leqslant \text{const}\, \frac{t-s}{\sigma^2} \int_s^t \mathrm{e}^{-\tilde{\rho}_+(t,v)^2/2\sigma^2} \psi_\searrow(v,s)\, \mathrm{d}v$$
$$\leqslant \text{const}\, \frac{t-s}{\sigma^3}\, \mathrm{e}^{-\text{const}\, \Delta_0^2/\sigma^2} \mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2}, \tag{4.51}$$

where the last line is obtained by the same argument as in Proposition 4.3, compare (4.32). This proves (4.44) for $n=1$, and for larger n the result follows easily by induction. ∎

As a direct consequence of Proposition 4.4, whenever $t-s<\sigma^2\mathrm{e}^{\mathrm{const}\,\Delta_0^2/\sigma^2}$, we can bound the sum in (4.43) by a geometric series, and conclude that

$$\frac{1}{\sigma}\,c_+(t,s)\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2}\leqslant q(t,s)\leqslant\frac{1}{\sigma}\,\bar{c}_+(t,s)\,\mathrm{e}^{-\rho_+(t,s)^2/2\sigma^2},\tag{4.52}$$

where $\bar{c}_+(t,s)=c_+(t,s)+\mathcal{O}((t-s)\,\sigma^{-2}(1+c_+(t,s))\,\mathrm{e}^{-\mathrm{const}\,\Delta_0^2/\sigma^2})$.

5. PROPERTIES OF THE EXIT LAW

The first-passage law of the process y_t at $+1$ can be expressed as a function of the first-passage density $\psi_-(s,0)$ to $1-\delta_1$ and $q(t,s)$ via (3.1) as

$$p_+(t)=\frac{\partial}{\partial t}\,\mathbb{P}^{0,-1}\{\tau_+\leqslant t\}=\int_0^t q(t,s)\,\psi_-(s,0)\,\mathrm{d}s.\tag{5.1}$$

From now on, we assume that $t\leqslant\sigma^3\mathrm{e}^{\beta\Delta_0^2/2\sigma^2}$ for some sufficiently small constant $\beta>0$, so that the error terms in (4.9) and (4.27) are at most of order $h(\sigma)=\sigma\mathrm{e}^{-\beta\Delta_0^2/2\sigma^2}$, and the one in (4.52) is at most of order $h(\sigma)(1+c_+(t,s))$.

5.1. Estimating the Integral by a Sum

In virtue of Propositions 4.1 and 4.3, and (4.52), the integral in (5.1) can be written as

$$\begin{aligned}p_+(t)=\frac{1}{\sigma^2}\int_0^t&[c_+(t,s)+\mathcal{O}(h(\sigma)[1+c_+(t,s)])]\\&\times c_-(s,0)\,\mathrm{e}^{-[\rho_+(t,s)^2+\rho_-(s,0)^2]/2\sigma^2}\,\mathrm{d}s.\end{aligned}\tag{5.2}$$

The exponent in (5.2) is of the form

$$\begin{aligned}\rho^{(0)}(t,s)^2&:=\rho_+(t,s)^2+\rho_-(s,0)^2\\&=\frac{\delta_1^2}{\hat{v}_+(t,s)}+\frac{(2-\delta_1)^2}{v_-(s,0)}\\&=\frac{\delta_1^2}{\hat{v}_+^{\mathrm{per}}(s)-\mathrm{e}^{-2\alpha(t,s)}\hat{v}_+^{\mathrm{per}}(t)}+\frac{(2-\delta_1)^2}{v_-^{\mathrm{per}}(s)-\mathrm{e}^{-2\alpha(s)}v_-^{\mathrm{per}}(0)},\end{aligned}\tag{5.3}$$

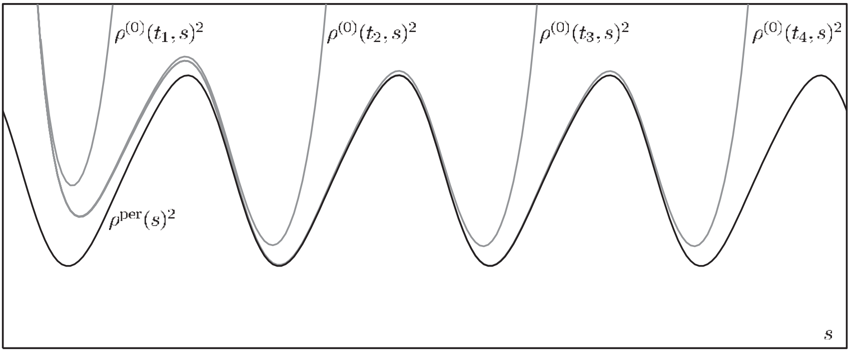

Fig. 7. The functions $\rho^{\text{per}}(s)^2$ and $\rho^{(0)}(t_i, s)^2$ for four different final times $t_1 < t_2 < t_3 < t_4$.

which we write for $0 \ll s \ll t$ as

$$\rho^{(0)}(t, s)^2 = \rho^{\text{per}}(s)^2 + \mathcal{O}(\mathrm{e}^{-2\alpha(t,s)}) + \mathcal{O}(\mathrm{e}^{-2\alpha(s)}) \tag{5.4}$$

with a periodic part

$$\rho^{\text{per}}(s)^2 = \frac{\delta_1^2}{\hat{v}_+^{\text{per}}(s)} + \frac{(2-\delta_1)^2}{v_-^{\text{per}}(s)}, \tag{5.5}$$

cf. Fig. 7. In Hypothesis H5, we assumed, for simplicity, that $\rho^{\text{per}}(s)$ has a unique minimum in the interval $[0, T)$, at some $s^\star \in [0, T)$ satisfying

$$\rho^{\text{per}}(s^\star) = \inf_{s \in [0,T)} \rho^{\text{per}}(s). \tag{5.6}$$

When the time interval $[0, t]$ includes many periods, $\rho^{(0)}(t, s)$ will have minima of comparable depths near all times $0 \ll s^\star + kT \ll t$, see Fig. 7. Other minima, which may exist for s near 0 and t are much shallower, and thus contribute less to the integral (5.2). The integral will be of the order $(t/T)\,\sigma \mathrm{e}^{-\rho^{\text{per}}(s^\star)^2/2\sigma^2}$. On the other hand, when t is not large enough, even the deepest minimum of $\rho^{(0)}(t, s)^2$ will be substantially larger than $\rho^{\text{per}}(s^\star)^2$, leading to a value of $p_+(t)$ which is orders of magnitude smaller than $(t/T)\,\sigma \mathrm{e}^{-\rho^{\text{per}}(s^\star)^2/2\sigma^2}$. The system is then still in its initial transient regime.

The transition between these regimes occurs when $\alpha(t) \simeq 2\,|\log \sigma|$. We first show that for times such that $\alpha(t)$ is smaller than $2\,|\log \sigma|$, the density $p_+(t)$ is much smaller than its "asymptotic" value $(t/\sigma T)\,\mathrm{e}^{-\rho^{\text{per}}(s^\star)^2/2\sigma^2}$.

Proposition 5.1. Assume that $\alpha(t) \leqslant 2\nu\,|\log\sigma|$ for some $\nu < 1$. Then

$$p_+(t) \leqslant \text{const}\,\frac{1}{\sigma^2}\,\mathrm{e}^{-L/\sigma^{2(1-\nu)}}\mathrm{e}^{-\rho^{\text{per}}(s^\star)^2/2\sigma^2}, \tag{5.7}$$

where $L > 0$ is a constant independent of σ, t, and ν.

Proof. From (5.3) we obtain

$$\rho^{(0)}(t,s)^2 \geqslant \rho^{\text{per}}(s)^2 + \delta_1^2\,\frac{\hat{v}_+^{\text{per}}(t)}{\hat{v}_+^{\text{per}}(s)^2}\,\mathrm{e}^{-2\alpha(t,s)} + (2-\delta_1)^2\,\frac{v_-^{\text{per}}(0)}{v_-^{\text{per}}(s)^2}\,\mathrm{e}^{-2\alpha(s)}. \tag{5.8}$$

The periodic functions $\hat{v}_+^{\text{per}}(s)$ and $v_-^{\text{per}}(s)$ being bounded above by $\bar{v}$, we have

$$\rho^{(0)}(t,s)^2 \geqslant \rho^{\text{per}}(s^\star)^2 + \beta_1^2\mathrm{e}^{-2\alpha(t,s)} + \beta_2^2\mathrm{e}^{-2\alpha(s)}, \tag{5.9}$$

where $\beta_1 = \delta_1\hat{v}_+^{\text{per}}(t)^{1/2}/\bar{v}$ and $\beta_2 = (2-\delta_1)\,v_-^{\text{per}}(0)^{1/2}/\bar{v}$. The right-hand side of (5.9) reaches its minimum when s satisfies $\beta_2\mathrm{e}^{-2\alpha(s)} = \beta_1\mathrm{e}^{-\alpha(t)}$, and has value

$$\rho^{\text{per}}(s^\star)^2 + 2\beta_1\beta_2\mathrm{e}^{-\alpha(t)} \geqslant \rho^{\text{per}}(s^\star)^2 + 2\beta_1\beta_2\sigma^{2\nu}. \tag{5.10}$$

We can now estimate the integral (5.2) by splitting it at times $t_1 = 1$ and $t_2 = t-1$. (If $t \leqslant 2$, the argument is even simpler.) The integral over $[t_1, t_2]$ can be bounded by using (5.10), while the integral on $[0, t_1]$ is small because for σ small enough, $c_-(s,0)\,\mathrm{e}^{-\rho_-(s,0)^2/2\sigma^2}$ reaches its maximum at t_1 and the remaining factor in the integrand is bounded. On $[t_2, t]$ the situation is similar. ∎

We now turn to the case where $t > 2\,|\log\sigma|/\lambda$. Assume that $t \in [nT, (n+1)\,T)$. If we can show that $\rho^{(0)}(t,s)$ has exactly one minimum s_k in $[kT, (k+1)\,T)$ for $1 \ll k \ll n$, and that this minimum is quadratic, the Laplace method will allow us to approximate the integral (5.2) by

$$\frac{1}{\sigma}\left(\sum_k \sqrt{\frac{4\pi}{\partial_{ss}(\rho^{(0)}(t,s_k)^2)}}\,c_+(t,s_k)\,c_-(s_k,0) + \text{error term}\right)\mathrm{e}^{-\rho^{(0)}(t,s_k)^2/2\sigma^2}. \tag{5.11}$$

Making this argument precise, we obtain the following result.

Proposition 5.2. Assume that $t \in [nT, (n+1)\,T)$ with $n\lambda T \geqslant 2\,|\log\sigma|$ and $n \geqslant 4$. Then

$$p_+(t) = \frac{1}{\sigma}\,C(s^\star)\,\frac{g(t)^2}{\hat{v}_+^{\text{per}}(t)}\,S(n,\sigma,t)[1+\mathcal{O}(\sigma)]\,\mathrm{e}^{-\rho^{\text{per}}(s^\star)^2/2\sigma^2}, \tag{5.12}$$

where $C(s^\star)$ is a constant given by

$$C(s^\star)=\frac{2(2-\delta_1)}{\delta_1}\frac{g(s^\star)^2}{\sqrt{\pi\partial_{ss}(\rho^{\mathrm{per}}(s^\star)^2)}}\frac{\hat{v}_+^{\mathrm{per}}(s^\star)^{1/2}}{v_-^{\mathrm{per}}(s^\star)^{3/2}}\left[1-\frac{v_-^{\mathrm{per}}(s^\star)}{2v^\star(s^\star)}\right], \quad (5.13)$$

and $S(n,\sigma,t)$ is the sum

$$S(n,\sigma,t)=\frac{\gamma(t)}{2}\sum_{k=1}^{n}\exp\left\{-2(n-k)\,\lambda T-\frac{1}{2\sigma^2}\left[\gamma_0\mathrm{e}^{-2k\lambda T}+\gamma(t)\,\mathrm{e}^{-2(n-k)\,\lambda T}\right]\right\}, \quad (5.14)$$

with

$$\gamma_0=(2-\delta_1)^2\,\mathrm{e}^{-2\alpha(s^\star)}\frac{v_-^{\mathrm{per}}(0)}{v_-^{\mathrm{per}}(s^\star)^2},$$
$$\gamma(t)=\delta_1^2\mathrm{e}^{-2\alpha(t,\,s^\star+nT)}\frac{\hat{v}_+^{\mathrm{per}}(t)}{\hat{v}_+^{\mathrm{per}}(s^\star)^2}. \quad (5.15)$$

Proof. We split the integral (5.2) at times $t_1=k_1T$ and $t_2=(n-k_1)\,T$, where $k_1<n/2$ will be chosen in such a way that for $t_1\leqslant s\leqslant t_2$, $\rho^{(0)}(t,s)^2$ has a minimum close to $s^\star+kT$ on each interval $I_k=[kT,(k+1)\,T)$, while the contributions of the integrals over $[0,t_1]$ and $[t_2,t]$ are negligible.

Take k_1 of the form $k_1=\lceil\nu\,|\log\sigma|/2\lambda T\rceil\vee 2$, with a parameter ν yet to be chosen. For $t_1\leqslant s\leqslant t_2$, one has $\mathrm{e}^{-2\alpha(s)}\leqslant\sigma^\nu$ and $\mathrm{e}^{-2\alpha(t,s)}\leqslant\sigma^\nu$. We first show that $\rho^{(0)}(t,s)^2$ has a quadratic minimum in each I_k, $k_1\leqslant k<n-k_1$. For this purpose, we write

$$\rho^{(0)}(t,s)^2=\frac{\delta_1^2}{\hat{v}_+^{\mathrm{per}}(s)-\mathrm{e}^{2\alpha(s,\,kT)}\gamma_1}+\frac{(2-\delta_1)^2}{v_-^{\mathrm{per}}(s)-\mathrm{e}^{-2\alpha(s,\,kT)}\gamma_2}, \quad (5.16)$$

where $\gamma_1=\mathrm{e}^{-2(n-k)\,\lambda T}\mathrm{e}^{-2\alpha(t,\,nT)}\hat{v}_+^{\mathrm{per}}(t)$ and $\gamma_2=\mathrm{e}^{-2k\lambda T}v_-^{\mathrm{per}}(0)$ are at most of order σ^ν. Were $\gamma_1=\gamma_2=0$, (5.16) would reduce to $\rho^{\mathrm{per}}(s)^2$, which has a unique minimum in I_k at $s^\star+kT$, the latter being quadratic. Hence, the implicit-function theorem applied to $\partial_s\rho^{(0)}(t,s)^2$ shows that for sufficiently small γ_1 and γ_2, $\rho^{(0)}(t,s)^2$ has a unique minimum in I_k at a time $s_k=s^\star+kT+\mathcal{O}(\sigma^\nu)$.

Expanding (5.16) into powers of γ_1 and γ_2 shows that

$$\rho^{(0)}(t,s_k)^2=\rho^{\mathrm{per}}(s^\star)^2+\gamma(t)\,\mathrm{e}^{-2(n-k)\,\lambda T}+\gamma_0\mathrm{e}^{-2k\lambda T}+\mathcal{O}(\sigma^{2\nu}), \quad (5.17)$$

for the coefficients γ_0 and $\gamma(t)$ given in (5.15). Evaluating the integral (5.2) restricted to the interval $[t_1, t_2]$ by the Laplace method yields

$$\frac{1}{\sigma}\sum_{k=k_1}^{n-k_1-1}\sqrt{\frac{4\pi}{\partial_{ss}(\rho^{\text{per}}(s_k)^2)}} \\ \times[c_+(t,s_k)+\mathcal{O}(h(\sigma)[1+c_+(t,s_k)])]\,c_-(s_k,0)[1+\mathcal{O}(\sigma^2)]\,\mathrm{e}^{-\rho^{(0)}(t,s_k)^2/2\sigma^2}, \tag{5.18}$$

where $\partial_{ss}\rho^{\text{per}}(s_k)=\partial_{ss}\rho^{\text{per}}(s^\star)+\mathcal{O}(\sigma^\nu)$ and

$$c_+(t,s_k)=\frac{\delta_1}{\sqrt{2\pi}}\frac{g(t)^2\,\mathrm{e}^{-2\alpha(t,s^\star+nT)}\mathrm{e}^{-2(n-k)\lambda T}}{\hat{v}_+^{\text{per}}(s^\star)^{3/2}}[1+\mathcal{O}(\sigma^\nu)]-\mathcal{O}(h(\sigma)), \tag{5.19}$$

$$c_-(s_k,0)=\frac{2-\delta_1}{\sqrt{2\pi}}\left[\frac{1}{v_-^{\text{per}}(s^\star)}-\frac{1}{2v^\star(s^\star)}\right]\frac{g(s^\star)^2}{v_-^{\text{per}}(s^\star)^{1/2}}[1+\mathcal{O}(\sigma^\nu+h(\sigma))]. \tag{5.20}$$

The error term $\mathcal{O}(\sigma^{2\nu})$ in the exponent (5.17) yields an error term $\mathcal{O}(\sigma^{2(\nu-1)})$ in the sum (5.18). For this reason, we choose $\nu=3/2$, so that all error terms are of order σ at most. This choice of ν is always possible, since $2k_1\leqslant 3\,|\log\sigma|/(2\lambda T)+2$ is smaller than n by our condition on $|\log\sigma|$. Note that the additive error term $\mathcal{O}(h(\sigma))$ in (5.19) can be incorporated into the sum $S(n,\sigma,t)$, as $nh(\sigma)\leqslant\mathcal{O}(th(\sigma))\leqslant\mathcal{O}(\sigma^4)\leqslant\mathcal{O}(\sigma^2)\,S(n,\sigma,t)$, where the last inequality will be proved in the next section.

We now turn to computing a bound for the integral (5.2) restricted to the interval $[0,t_1]$. We first consider the case $|\log\sigma|\geqslant 5\lambda T$, in which $k_1\geqslant 4$ and $\mathrm{e}^{-2\alpha(t_1)}=\mathrm{e}^{-2k_1\lambda T}\geqslant\sigma^{8/5}$. With the help of (5.9), the integral can be estimated by $\exp\{-\rho^{\text{per}}(s^\star)^2/2\sigma^2\}$ times its maximal value which itself is bounded by

$$\frac{\text{const}}{t_1^{1/2}\sigma^2}\exp\left\{-\frac{\beta_2^2}{2\sigma^2}\,\mathrm{e}^{-2\alpha(t_1)}\right\}=\mathcal{O}\left(\frac{1}{\sigma^2\,|\log\sigma|^{1/2}}\exp\left\{-\frac{\beta_2^2}{2\sigma^{2/5}}\right\}\right). \tag{5.21}$$

Hence the first part of the integral is exponentially small compared to the integral itself. A similar estimate shows that the integral over $[t_2,t]$ is small.

In the case $|\log\sigma|<5\lambda T$, the location of the minima has to be estimated with more care. In this case, $\mathrm{e}^{-2\lambda T}\leqslant\sigma^{2/5}$, so that the same argument as before shows that $\rho^{(0)}(t,s)^2$ has a minimum in each of the intervals I_1, I_2, and I_3, and possibly also in I_0. An examination of $\partial_s\rho^{(0)}(t,s)^2$ shows that these minima are actually located in $s_k=s^\star+kT+\mathcal{O}(\mathrm{e}^{-2\alpha(s^\star+kT)})$,

$k=0, 1, 2, 3$. We set $\kappa=\alpha(s^\star)/\lambda T$, $0\leqslant\kappa<1$, and distinguish between two cases:

- If $4(k+\kappa)\,\lambda T\geqslant 3\,|\log\sigma|$, then $\rho^{(0)}(t, s_k)^2=\rho^{\rm per}(s^\star)^2+\gamma_0\mathrm{e}^{-2k\lambda T}+\mathcal{O}(\sigma^3)$, and the integral between 0 and s_k is exponentially small.
- If $4(k+\kappa)\,\lambda T<3\,|\log\sigma|$, then $\mathrm{e}^{-2\alpha(s_k)}\geqslant\sigma^{3/2}$, and the error term only leads to an error of order σ in the prefactor.

The intervals I_k, $n-4\leqslant k\leqslant n-1$, are treated in a similar way.

The same type of arguments shows that the error made by extending the sum in (5.18) to all k between 1 and n is also negligible. ▮

5.2. Properties of the Sum

To complete the proof of Theorem 2.3, we will show that the sum $S(n, \sigma, t)$, defined in (5.14), is close to a periodic function of $|\log\sigma|$. In this section, we always assume that $t\in[nT, (n+1)\,T)$ and that $n\lambda T\geqslant 2\,|\log\sigma|$ as well as $n\geqslant 4$. To highlight the σ-dependence of S, it is convenient to set $\sigma=\mathrm{e}^{-\eta}$ and $S(n, \sigma, t)=\sigma^2\tilde{S}(n, \eta, t)$. Further introducing the notations

$$\theta_0=-\tfrac{1}{2}\log\gamma_0, \qquad \bar{\theta}(t)=-\tfrac{1}{2}\log\gamma(t), \tag{5.22}$$

and changing the summation index from k to $\ell=n-k$, allows to write the sum in compact form as

$$\tilde{S}(n, \eta, t)=\sum_{\ell=0}^{n-1} A(\ell\lambda T-\eta+\bar{\theta}(t))\, B((n-\ell)\,\lambda T-\eta+\theta_0), \tag{5.23}$$

with

$$A(x)=\tfrac{1}{2}\exp\{-2x-\tfrac{1}{2}\,\mathrm{e}^{-2x}\}, \tag{5.24}$$

$$B(x)=\exp\{-\tfrac{1}{2}\,\mathrm{e}^{-2x}\}. \tag{5.25}$$

The function $A(x)$ decays like e^{-2x} as $x\to+\infty$, and like $\exp\{-\frac{1}{2}\,\mathrm{e}^{2\,|x|}\}$ as $x\to-\infty$. It reaches its maximal value e^{-1} when $\mathrm{e}^{-2x}=2$. The function $B(x)$ is monotonously increasing. It decays like $\exp\{-\frac{1}{2}\,\mathrm{e}^{2\,|x|}\}$ as $x\to-\infty$ and approaches 1 as $x\to+\infty$.

Since $\gamma(t)$ involves the location of $t\in[nT, (n+1)\,T)$ relatively to $s^\star+nT$ (see (5.15)), $\bar{\theta}(t)$ is a (right-continuous) periodic saw-tooth function, making a jump of $-\lambda T$ at each integer multiple of T. We can thus write it as

$$\bar{\theta}(t)=\theta(t)-\lambda T\left\lfloor\frac{t}{T}\right\rfloor, \tag{5.26}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x, and $\theta(t)$ is a continuous function given by

$$\theta(t) = \alpha(t, s^\star) - \frac{1}{2}\log \hat{v}_+^{\text{per}}(t) - \log \frac{\delta_1}{\hat{v}_+^{\text{per}}(s^\star)}. \tag{5.27}$$

Note that $\theta(t+T) = \theta(t) + \lambda T$.

It is easy to show that $\tilde{S}(n, \eta + \lambda T, t) - \tilde{S}(n-2, \eta, t-2T)$ is exponentially small (just shift the index of summation and show that the boundary terms are exponentially small). The following result gives a more precise characterization of $\tilde{S}(n, \eta, t)$ by showing that it is actually close to a periodic function of $\eta - \theta(t)$.

Proposition 5.3. Assume that $n\lambda T \geqslant 2\eta$. Then

$$\tilde{S}(n, \eta, t) = \hat{S}(\eta, t)[1 + \mathcal{O}(\sigma^\mu)], \tag{5.28}$$

where $\mu = \mu(n, \eta) = (n\lambda T - 2\eta)/\eta$ and

$$\hat{S}(\eta, t) = \sum_{\ell=-\infty}^{\infty} A(\ell\lambda T - \eta + \theta(t)). \tag{5.29}$$

Proof. We split the sum $\tilde{S}(n, \eta, t)$ into two parts. For $0 \leqslant \ell \leqslant n/2$, we use the fact that

$$\begin{aligned} 1 &\geqslant B((n-\ell)\,\lambda T - \eta + \theta_0) \geqslant B\left(\frac{n}{2}\lambda T - \eta + \theta_0\right) \\ &= \exp\left\{-\frac{\gamma_0}{2}\,\mathrm{e}^{2\eta - n\lambda T}\right\} = 1 - \mathcal{O}(\sigma^\mu). \end{aligned} \tag{5.30}$$

Hence replacing B by 1 in the sum for these values of ℓ only yields a multiplicative error of order $1 - \mathcal{O}(\sigma^\mu)$. For $\ell > n/2$, it is obvious that

$$A(\ell\lambda T - \eta + \bar{\theta}(t)) \leqslant \frac{\gamma(t)}{2}\,\mathrm{e}^{-2\ell\lambda T + 2\eta}. \tag{5.31}$$

Bounding B by 1 allows to bound the sum over $\ell = \lceil n/2 \rceil, \dots, n$ by the geometric series

$$\sum_{\ell=\lceil n/2 \rceil}^{n} A(\ell\lambda T - \eta + \bar{\theta}(t)) \leqslant \frac{\gamma(t)}{2} \sum_{\ell=\lceil n/2 \rceil}^{n} \mathrm{e}^{-2\ell\lambda T + 2\eta} = \mathcal{O}(\sigma^\mu). \tag{5.32}$$

Thus the main contribution to $\tilde{S}$ stems from $\ell \in \{1,\dots,\lfloor n/2 \rfloor\}$, and so does the main contribution to $\sum_{\ell=0}^{\infty} A(\ell\lambda T - \eta + \theta(t))$. It remains to check that the contribution of negative ℓ to $\hat{S}$ is small. Comparing that sum with an integral shows that it is in fact of the order $\mathrm{e}^{-\gamma(t)/2\sigma^2}$. Finally, replacing $\bar{\theta}(t)$ by $\theta(t)$ only results in a shift of the summation index. ∎

The function $\hat{S}(\eta, t)$ is clearly periodic in $\eta - \theta(t)$ with period λT. Let us thus write $\hat{S}(\eta, t) = P((\eta - \theta(t))/\lambda T)$, where $P(x) > 0$ is periodic with period 1. It remains to compute the Fourier series of $P(x)$.

Proposition 5.4. The periodic function $P(x)$ admits the Fourier series

$$P(x) = \sum_{q=-\infty}^{\infty} \hat{P}(q)\, \mathrm{e}^{2\pi i q x}, \tag{5.33}$$

where the qth Fourier coefficient is given in terms of the Euler Gamma function by

$$\hat{P}(q) = \frac{1}{2\lambda T} \frac{1}{2^{\pi i q/\lambda T}}\, \Gamma\left(1 - \frac{\pi i q}{\lambda T}\right). \tag{5.34}$$

Proof. We have

$$\begin{aligned} \hat{P}(q) &= \int_0^1 P(x)\, \mathrm{e}^{-2\pi i q x}\, \mathrm{d}x \\ &= \sum_{\ell=-\infty}^{\infty} \int_0^1 A((\ell - x)\, \lambda T)\, \mathrm{e}^{-2\pi i q x}\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} A(\lambda T x)\, \mathrm{e}^{2\pi i q x}\, \mathrm{d}x. \end{aligned} \tag{5.35}$$

Replacing A by its definition and using the change of variable $z = \mathrm{e}^{-2\lambda T x}/2$ yields the result. ∎

APPENDIX A

Let $v(t)$ be continuously differentiable on $[0, \infty)$ and satisfy $v(0) = 0$ and $v'(t) \geqslant v_0 > 0$ for all $t \geqslant 0$. As before, we denote by W_t a standard Brownian motion. Consider the Gaussian process

$$z_t = \sigma \int_0^t \sqrt{v'(s)}\, \mathrm{d}W_s, \tag{A.1}$$

whose variance is $\sigma^2 v(t)$. We will consider z_t as a Markov process and introduce the notation $\mathbb{P}^{s,x}\{z_t \in \cdot\} = \mathbb{P}\{z_t \in \cdot \mid z_s = x\}$, $t > s$, for its transition probabilities. Their densities are given by

$$\begin{aligned} y \mapsto f(t, y \mid s, x) &:= \frac{\partial}{\partial y} \mathbb{P}^{s,x}\{z_t \leqslant y\} \\ &= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi v(t,s)}} \mathrm{e}^{-(y-x)^2/2\sigma^2 v(t,s)} \end{aligned} \tag{A.2}$$

where $v(t,s) = v(t) - v(s)$.

Let $d(t)$ be continuously differentiable on $[0, \infty)$ and satisfy $d(0) > 0$. The object of the "level-crossing problem" is to determine the density $\psi(t)$ of the first-passage time $\tau = \inf\{s > 0 : z_s > d(s)\}$ (which we will call " first-passage density of z_t to $d(t)$"). This problem has for instance been studied in refs. 12, 13, 23, and 24. The aim of this appendix is to establish expressions for $\psi(t)$ useful in our particular situation.

As D. Williams in the appendix to ref. 13, we will use two integral equations satisfied by $\psi(t)$. Let $F(t) = f(t, d(t) \mid 0, 0)$ denote the value of the density of z_t at $d(t)$ and let $F(t \mid s) = f(t, d(t) \mid s, d(s))$ denote the transition density at $y = d(t)$ for paths starting at time s in $x = d(s)$. The Markov property enables us to write

$$F(t) = \int_0^t F(t \mid s)\, \psi(s)\, \mathrm{d}s. \tag{A.3}$$

The second integral equation satisfied by $\psi(t)$ is

$$\psi(t) = b_0(t)\, F(t) - \int_0^t \tilde{b}(t,s)\, F(t \mid s)\, \psi(s)\, \mathrm{d}s, \tag{A.4}$$

where

$$b_0(t) = v'(t) \left[\frac{d(t)}{v(t)} - \frac{d'(t)}{v'(t)} \right] = -v(t) \frac{\partial}{\partial t} \left(\frac{d(t)}{v(t)} \right), \tag{A.5}$$

$$\tilde{b}(t,s) = v'(t) \left[\frac{d(t,s)}{v(t,s)} - \frac{d'(t)}{v'(t)} \right] = -v(t,s) \frac{\partial}{\partial t} \left(\frac{d(t,s)}{v(t,s)} \right), \tag{A.6}$$

with $d(t,s) = d(t) - d(s)$. In the particular case of a standard Brownian motion, that is for $\sigma = 1$ and $v(t) \equiv t$, Eq. (A.4) has been established in ref. 13, Appendix by D. Williams. The general case is easily obtained from the fact that $z_t = \sigma W_{v(t)}$ in distribution.

Equation (A.4) suggests that the first-passage density can be written in the form

$$\psi(t) = \frac{1}{\sigma}\, c(t)\, \mathrm{e}^{-d(t)^2/2\sigma^2 v(t)}, \tag{A.7}$$

where $c(t)$ is a subexponential prefactor. In fact, the following bound on $c(t)$ follows immediately from (A.3) and (A.4).

Lemma A.1. Let $c_0(t) = b_0(t)/\sqrt{2\pi v(t)}$. Then (A.7) holds with

$$|c(t) - c_0(t)| \leqslant \frac{1}{\sqrt{2\pi v(t)}} \sup_{0 \leqslant s \leqslant t} |\tilde{b}(t, s)|. \tag{A.8}$$

Remark A.2. Note that Lemma A.1 does not require $v'(t)$ to be bounded away from zero as t varies.

If, for instance, $v(t)$ and $d(t)$ are twice continuously differentiable, then $s \mapsto \tilde{b}(t, s)$ is easily seen to be bounded, and thus $c(t)$ behaves like $v(t)^{-3/2}$ near $t = 0$.

In ref. 13, an expansion of $c(t)$ is constructed, which is shown to converge for all times t, under a convexity assumption on the boundary $d(t)$. Taking advantage of the fact that σ is a small parameter, we can control the convergence of this expansion under a milder assumption on $d(t)$, on a finite, but exponentially long time interval. Writing $\tilde{c}(t, s) = \tilde{b}(t, s)/\sqrt{2\pi v(t, s)}$, we see from (A.4) that $c(t)$ must be a fixed point of the operator

$$(\mathscr{T} c)(t) = c_0(t) - \frac{1}{\sigma} \int_0^t \tilde{c}(t, s)\, c(s)\, \mathrm{e}^{-r(t, s)/2\sigma^2}\, \mathrm{d}s, \tag{A.9}$$

where

$$r(t, s) = \frac{d(s)^2}{v(s)} - \frac{d(t)^2}{v(t)} + \frac{d(t, s)^2}{v(t, s)} = \frac{v(t)\, v(s)}{v(t, s)} \left[\frac{d(s)}{v(s)} - \frac{d(t)}{v(t)} \right]^2. \tag{A.10}$$

Remark A.3. The exponent $r(t, s)$ is nonnegative and vanishes for $s = t$. If $r(t, s)$ does not vanish anywhere else, then the main contribution to the integral in (A.9) comes from s close to t. In the generic case $\partial_s r(t, t) \neq 0$, the integral is at most of order σ^2. If the functions involved are sufficiently smooth, one easily sees that the integral is of order σ^3. If $r(t, s)$ vanishes in a quadratic minimum in $s = t$ or elsewhere, then the integral is at most of order σ.

It is well known that the most probable path reaching z at time t is represented by a straight line in the (v, z)-plane. Thus $r(t, s)$ vanishes for some $s \neq t$ if and only if the most probable path reaching $d(t)$ has already reached $d(s)$. In that case, there exists a time $u \in (0, t)$ such that the tangent to the curve $(v(s), d(s))_{s \geqslant 0}$ at $(v(u), d(u))$ goes through the origin, i.e., $(d(u)/v(u))' = 0$. This situation can be excluded under a convexity assumption on $d(t)$, which is equivalent to Hypothesis H3.

The following lemma establishes the existence and some properties of a fixed point of (A.9) under hypotheses tailored to our situation. We will employ it in Section 4.1 with $v(t) = \int_0^t \mathrm{e}^{2\alpha(s)} g(s)^2 \, \mathrm{d}s$ and $d(t) = (2-\delta_1) \, \mathrm{e}^{\alpha(s)}$.

Lemma A.4. Assume that there are constants $\Delta, M_1, M_2 > 0$ such that the conditions

$$d(s) \, v'(s) - v(s) \, d'(s) \geqslant \Delta v'(s)(1+\sqrt{v(s)}), \tag{A.11}$$

$$|\tilde{c}(t, s)| \leqslant M_1, \tag{A.12}$$

$$M_2 v'(s) \geqslant 1 + v(s), \tag{A.13}$$

hold for all $0 \leqslant s \leqslant t$. Then (A.7) holds with a prefactor $c(t)$ satisfying

$$|c(t) - c_0(t)| \leqslant \frac{\varepsilon}{1-\varepsilon} \frac{1+v(t)^{3/2}}{v(t)^{3/2}} \sup_{0 \leqslant s \leqslant t} \left| \frac{v(s)^{3/2}}{1+v(s)^{3/2}} c_0(s) \right|, \tag{A.14}$$

whenever

$$\varepsilon := 2M_1 \left(\frac{\mathrm{e}^{-\Delta^2/4\sigma^2}}{\sigma} t + \frac{4M_2 \sigma}{\Delta^2} \right) < 1. \tag{A.15}$$

Proof. We shall prove that $\mathscr{T}$ is a contraction on the Banach space $\mathscr{X}$ of continuous functions c: $[0, t] \to [0, \infty)$, equipped with the norm

$$\|c\| = \sup_{0 \leqslant s \leqslant t} \left| \frac{v(s)^{3/2}}{1+v(s)^{3/2}} c(s) \right|. \tag{A.16}$$

For any two functions $c_1, c_2 \in \mathscr{X}$, we have by (A.12)

$$|\mathscr{T} c_2(t) - \mathscr{T} c_1(t)| \leqslant \|c_2 - c_1\| \frac{M_1}{\sigma} \int_0^t \frac{1+v(s)^{3/2}}{v(s)^{3/2}} \mathrm{e}^{-r(t,s)/2\sigma^2} \, \mathrm{d}s. \tag{A.17}$$

Using Assumption (A.11), we obtain

$$\frac{d(s)}{v(s)}-\frac{d(t)}{v(t)}=\int_s^t \frac{d(u)\,v'(u)-v(u)\,d'(u)}{v(u)^2}\,\mathrm{d}u$$
$$\geqslant \Delta\left[\frac{v(t,s)}{v(t)\,v(s)}+2\frac{\sqrt{v(t)}-\sqrt{v(s)}}{\sqrt{v(t)\,v(s)}}\right], \tag{A.18}$$

and thus

$$r(t,s)\geqslant \Delta^2\left[\frac{v(t,s)}{v(t)\,v(s)}+4\frac{(\sqrt{v(t)}-\sqrt{v(s)})^2}{v(t,s)}\right]\geqslant \Delta^2\frac{v(t,s)}{v(t)}\left[1+\frac{1}{v(s)}\right]. \tag{A.19}$$

For the sake of brevity, we restrict our attention to the case $v(t)>2$. We split the integral in (A.17) at times s_1 and s_2 defined by $v(s_1)=1$ and $v(s_2)=v(t)/2$. By (A.13), the integral on the first interval is bounded by

$$2\int_0^{s_1}\frac{1}{v(s)^{3/2}}\,\mathrm{e}^{-\Delta^2/4\sigma^2 v(s)}\,\mathrm{d}s\leqslant\frac{4M_2\sigma}{\Delta}\int_{\Delta^2/4\sigma^2}^{\infty}\frac{\mathrm{e}^{-y}}{\sqrt{y}}\,\mathrm{d}y\leqslant\frac{8M_2\sigma^2}{\Delta^2}\,\mathrm{e}^{-\Delta^2/4\sigma^2}. \tag{A.20}$$

The second part of the integral is smaller than $2t\mathrm{e}^{-\Delta^2/4\sigma^2}$ because $r(t,s)\geqslant\Delta^2/2$ for $s_1<s<s_2$, while the last part is bounded by

$$\int_{s_2}^{t}\mathrm{e}^{-\Delta^2 v(t,s)/2\sigma^2 v(t)}\,\mathrm{d}s\leqslant\frac{2^{3/2}M_2}{v(t)}\int_{s_2}^{t}v'(s)\,\mathrm{e}^{-\Delta^2 v(t,s)/2\sigma^2 v(t)}\,\mathrm{d}s\leqslant\frac{2^{5/2}M_2\sigma^2}{\Delta^2}. \tag{A.21}$$

This shows that $\mathscr{T}$ is a contraction with contraction constant ε, and the result follows by bounding $\|c-c_0\|=\|\mathscr{T}^n c-\mathscr{T}0\|$ by a geometric series. Here 0 denotes the function which is zero everywhere. ∎

Corollary A.5. Let the assumptions of Lemma A.4 be satisfied and assume in addition that there exists a constant $M_3>0$ such that

$$d(t)\,v'(t)-v(t)\,d'(t)\leqslant M_3(1+v(t)^{3/2})\qquad\text{for all}\quad t\geqslant 0. \tag{A.22}$$

Then

$$c_0(t)\left[1-\frac{\varepsilon}{1-\varepsilon}\frac{M_2M_3}{\Delta}\right]\leqslant c(t)\leqslant c_0(t)\left[1+\frac{\varepsilon}{1-\varepsilon}\frac{M_2M_3}{\Delta}\right] \tag{A.23}$$

holds for all $t>0$ such that $\varepsilon=\varepsilon(t)<1$, where ε is defined by (A.15).

Proof. The proof follows directly from the bounds (A.22) and

$$\frac{1+v(t)^{3/2}}{v(t)^{3/2}} \frac{1}{c_0(t)} \leqslant \frac{\sqrt{2\pi}\, M_2}{\Delta}, \tag{A.24}$$

the latter being a consequence of (A.11) and (A.13). ∎

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